Lesson 10

Transforming 3D Integrals

Example 1: Triple Integrals to Compute Volume

Recall that in previous chapters we could find the length of an interval I by computing $\int_I dx$ or the area of a region R by computing $\iint_R dA$.

It follows that we can compute the volume of a 3-dimensional region R by calculating $\iiint_R 1\, dV$.

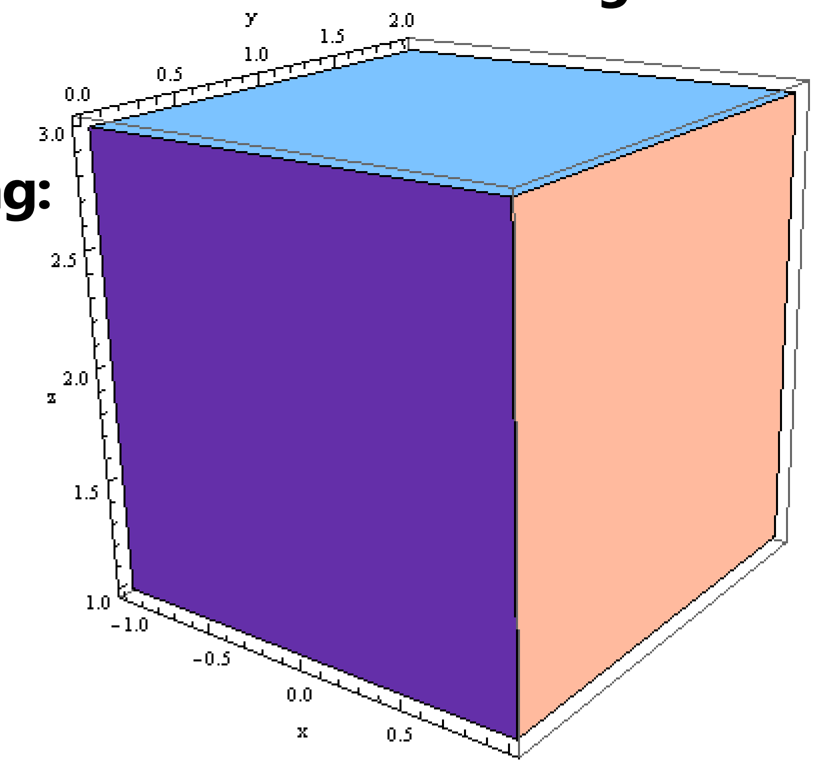
For this cube, the calculation is not exciting:

$$\int_1^3 \int_0^2 \int_{-1}^1 1\, dx\, dy\, dz = \int_1^3 \int_0^2 \left[x\right]_{x=-1}^{x=1} dy\, dz$$

$$= \int_1^3 \left[2y\right]_{y=0}^{y=2} dz$$

$$= \left[4z\right]_{z=1}^{z=3}$$

$$= 8$$

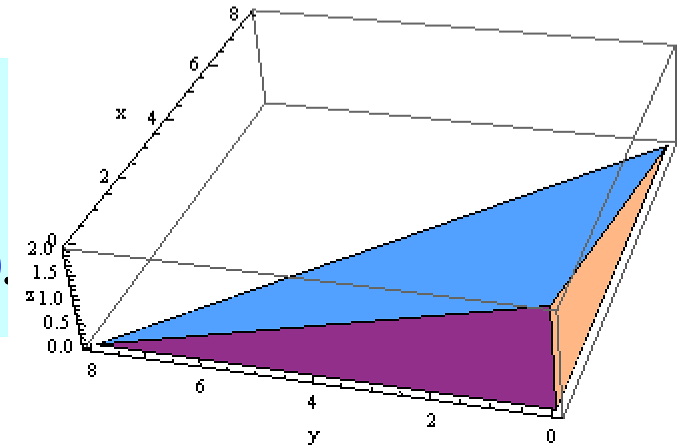


$-1 \le x \le 1,\ 0 \le y \le 2,\ 1 \le z \le 3$

Example 2: Triple Integrals to Compute Volume

We can spice things up a bit. We can find the volume of the region above the xy-plane, the xz-plane, the yz-plane, and below the plane given by $x + y + 4z = 8$:

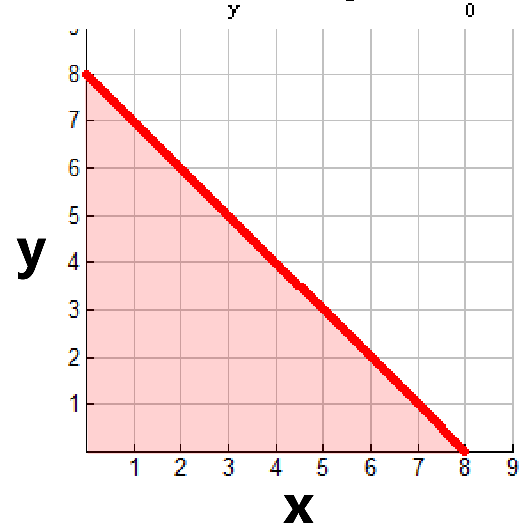
If $x + y + 4z = 8$, then $z = \dfrac{8 - x - y}{4}$. So z-top is the plane $z = \dfrac{8 - x - y}{4}$ and z-bottom is the plane $z = 0$.



Find the intersection between $z = \dfrac{8 - x - y}{4}$ and $z = 0$: We get the line $y = 8 - x$, which is our y-top. Our y-bottom is the line $y = 0$.

Finally, integrate x from x = 0 to x = 8.

$$\int_0^8 \int_0^{8-x} \int_0^{\frac{8-x-y}{4}} 1 \, dz \, dy \, dx$$



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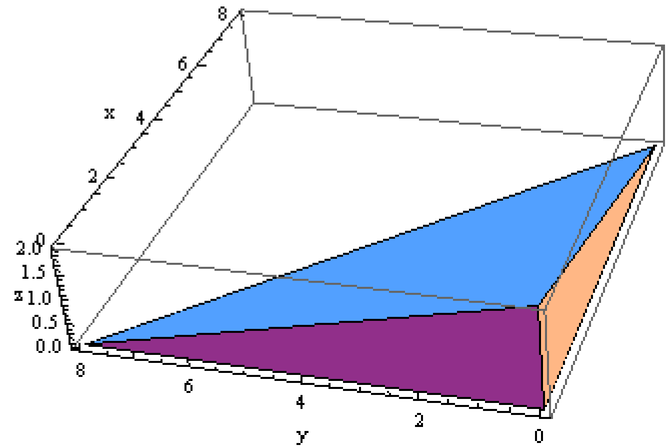


$$\int_0^8 \int_0^{8-x} \int_0^{\frac{8-x-y}{4}} 1\, dz\, dy\, dx = \frac{1}{4}\int_0^8 \int_0^{8-x} (8 - x - y)\, dy\, dx$$

$$= \frac{1}{4}\int_0^8 \left[8y - xy - \frac{y^2}{2} \right]_{y=0}^{y=8-x} dx$$

$$= \frac{1}{4}\int_0^8 \left(8(8-x) - x(8-x) - \frac{(8-x)^2}{2} \right) dx$$

$$= \frac{64}{3}$$

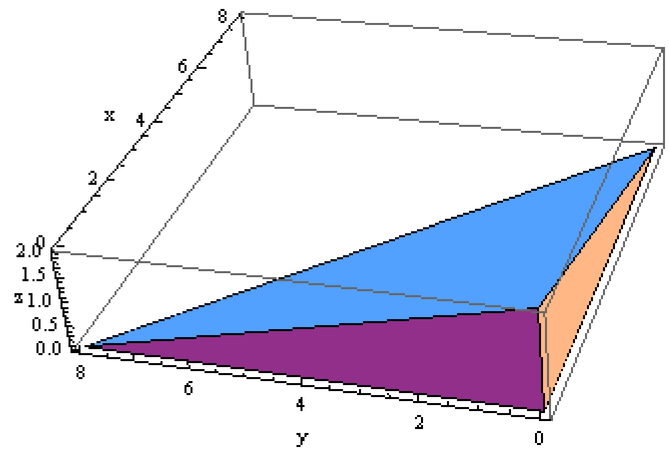
$$V_{pyramid} = \frac{1}{3} A_{base} h$$

$$= \frac{1}{3} \bullet 32 \bullet 2$$

$$= \frac{64}{3}$$

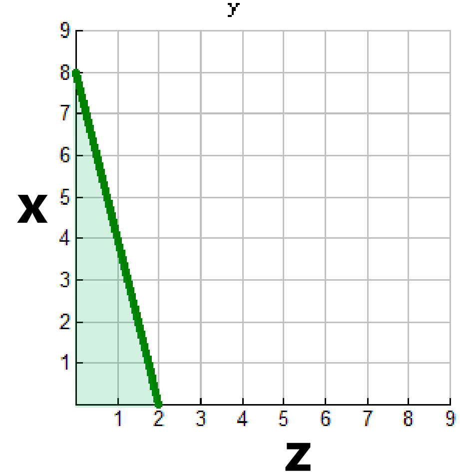
Example 3: Changing the Order of Integration

Repeat Example 2 by computing $\iiint\limits_{R} dy\,dx\,dz$ where R is the region above the xy-plane, the xz-plane, the yz-plane, and below the plane given by $x + y + 4z = 8$



If $x + y + 4z = 8$, then $y = 8 - x - 4z$. So y-top is $8 - x - 4z$ and y-bottom is the plane $y = 0$.

$x + y + 4z = 8$ and $y = 0$ intersect at the line $x = 8 - 4z$ (our x-top). Our x-bottom is the line $x = 0$.

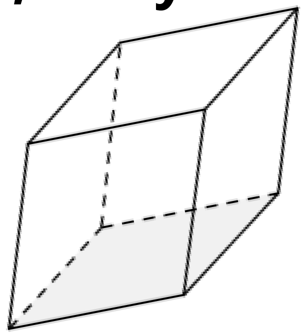


Finally, integrate z from $z = 0$ to $z = 2$.

$$\int_0^2 \int_0^{8-4z} \int_0^{8-x-4z} 1\,dy\,dx\,dz = \frac{64}{3}$$

Example 4: 3D-Change of Variables

Compute the volume of the parallelepiped described by the region R_{xyz} between the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $y = -2x + 3$.



First, pick a clever change of variables:

$u = z - 3x$, and let u run from 0 to 2

$v = y - x$, and let v run from 0 to 4

$w = y + 2x$, and let w run from 0 to 3

Our integral is much more manageable now:

$$\iiint_{R_{xyz}} 1\,dx\,dy\,dz = \iiint_{R_{uvw}} \left|V_{xyz}(u,v,w)\right|\,du\,dv\,dw$$

$$\int_0^3\int_0^4\int_0^2 \left|V_{xyz}(u,v,w)\right|\,du\,dv\,dw$$

$$V_{xyz}(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The Volume Conversion Factor:

$$\iiint_{R_{xyz}} 1\,dx\,dy\,dz = \iiint_{R_{uvw}} \left|V_{xyz}(u,v,w)\right|\,du\,dv\,dw$$

Let T(u, v, w) be a transformation from uvw-space to xyz-space.

That is, $T(u,v,w) = \left(T_1(u,v,w), T_2(u,v,w), T_3(u,v,w)\right) = (x(u,v,w), y(u,v,w), z(u,v,w))$.

Then $V_{xyz}(u,v,w) = \begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & \frac{\partial T_3}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & \frac{\partial T_3}{\partial v} \\ \frac{\partial T_1}{\partial w} & \frac{\partial T_2}{\partial w} & \frac{\partial T_3}{\partial w} \end{vmatrix}$ Or $V_{xyz}(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$

Note: Just like $A_{xy}(u,v)$, we need $V_{xyz}(u,v,w)$ to be positive. Hence, the absolute value bars in the formula above.

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$$u = z - 3x$$
$$v = y - x$$
$$w = y + 2x$$

$$\Rightarrow$$

$$x = \frac{w - v}{3}$$
$$y = \frac{2v + w}{3}$$
$$z = u + w - v$$

$$V_{xyz}(u,v,w) = \begin{vmatrix} 0 & 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

$$= -\frac{1}{3}$$

Use $\frac{1}{3}$.

$$\int_0^3\int_0^4\int_0^2 \left|V_{xyz}(u,v,w)\right| \, du\, dv\, dw = \int_0^3\int_0^4\int_0^2 \frac{1}{3} \, du\, dv\, dw$$

$$= 8$$

Example 4: 3D-Change of Variables

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No need to study this, it's just nice to verify this works...

Check :

For a parallelepiped generated by three intersecting vectors X, Y and Z,

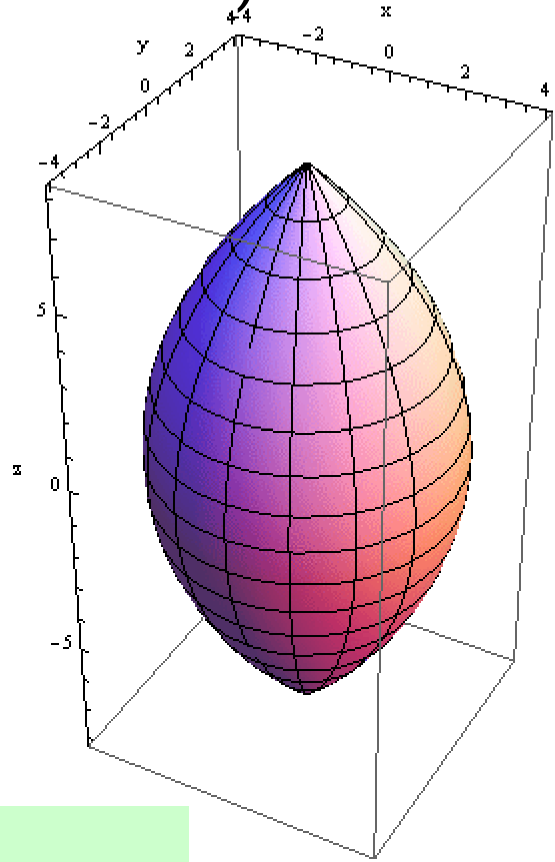
$$V_{parallelepiped} = (X \times Y) \bullet Z$$

Parallelepiped is generated by vectors (1,1,3), $\left(-\frac{4}{3}, \frac{8}{3}, -4\right)$, and (0,0,2).

$$V_{parallelepiped} = \left((1,1,3) \times \left(-\frac{4}{3}, \frac{8}{3}, -4\right)\right) \bullet (0,0,2) = 8$$

Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described by the parametric equation $F(s,t) = (0,0,4s) + \left(4 - s^2\right)\left(\cos(t), \sin(t), 0\right)$ for $-2 \le s \le 2$ and $0 \le t \le 2\pi$.



First, fill in the football:

$$F(r,s,t) = (0,0,4s) + r\left(4 - s^2\right)\left(\cos(t), \sin(t), 0\right)$$

$$-2 \le s \le 2$$

$$0 \le t \le 2\pi$$

$$0 \le r \le 1$$

Change of variables:

$$x(r,s,t) = r\left(4 - s^2\right)\cos(t)$$

$$y(r,s,t) = r\left(4 - s^2\right)\sin(t)$$

$$z(r,s,t) = 4s$$

$$\int_0^{2\pi}\int_{-2}^{2}\int_0^1 \left|V_{xyz}(r,s,t)\right| \, dr \, ds \, dt$$

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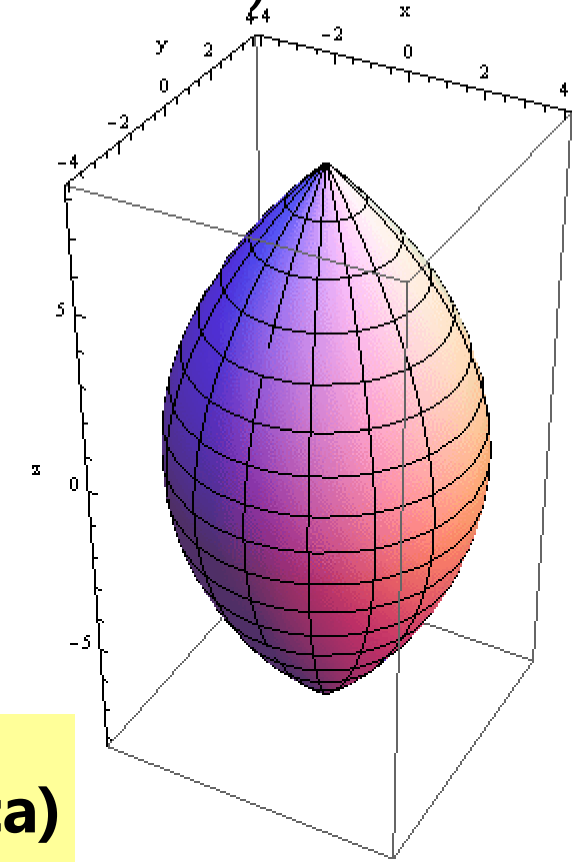
Find the volume of the "football" whose outer skin is described by the parametric equation $F(s,t) = (0,0,4s) + (4 - s^2)\big(\cos(t), \sin(t), 0\big)$ for $-2 \le s \le 2$ and $0 \le t \le 2\pi$.



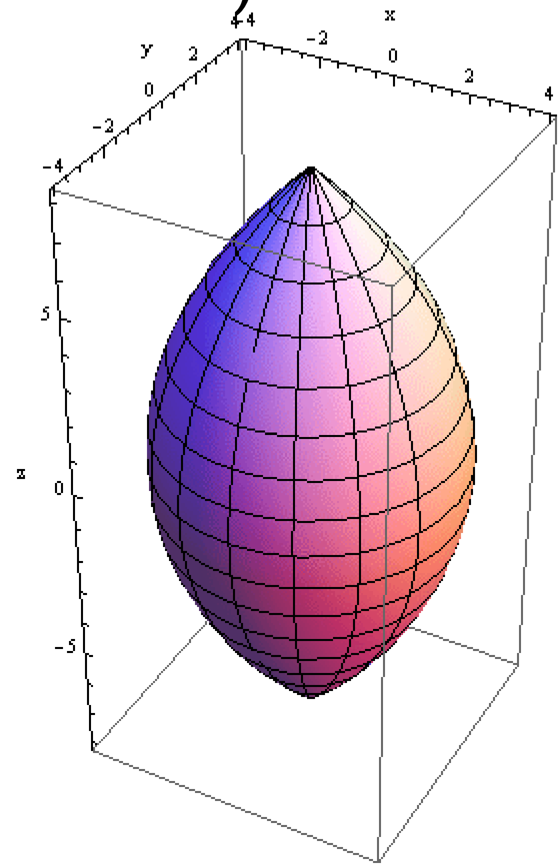
$$V_{xyz}(r,s,t) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix}$$

$$= -4(16r - 8rs^2 + rs^4) \quad \text{(Mathematica)}$$

$$\int_0^{2\pi}\int_{-2}^{2}\int_0^{1} \left|V_{xyz}(r,s,t)\right| \, dr\, ds\, dt = \frac{2048}{15}\pi \quad \text{(Mathematica)}$$

Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described by the parametric equation $F(s,t) = (0,0,4s) + (4 - s^2)\left(\cos(t), \sin(t), 0\right)$ for $-2 \le s \le 2$ and $0 \le t \le 2\pi$.



Check using solids of revolution:

$$\pi \int_{-8}^{8} \left(4 - \left(\frac{x}{4}\right)^2\right)^2 dx = \frac{2048}{15}\pi$$

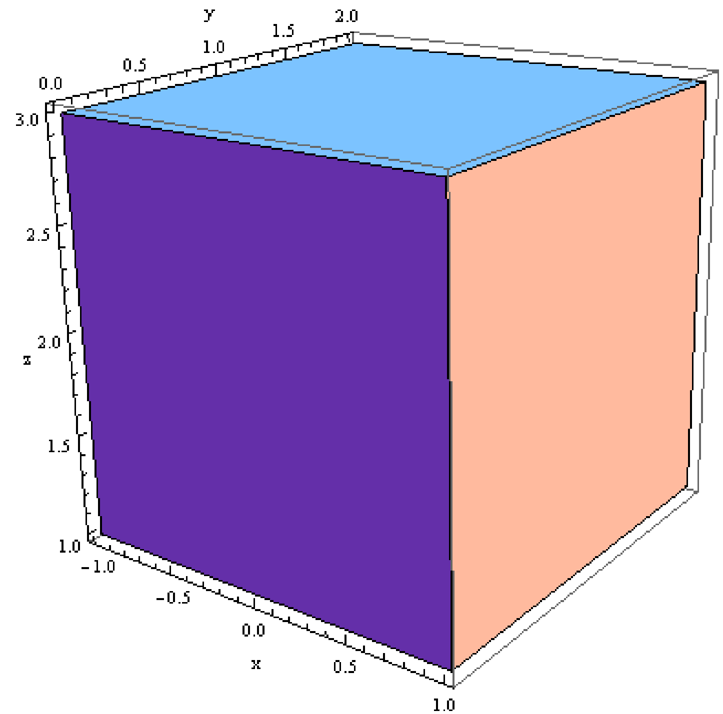
Example 6: Beyond Volume Calculations

It is easy to see that $\iiint\limits_R dV$ computes the volume of a solid. But it's harder to interpret $\iiint\limits_R f(x,y,z)\,dV$ since f(x,y,z) lives in 4 dimensions.

We need an example to keep referring back to to give us some intuition.

A cube of varying density has its density at each point (x, y, z) described by $f(x,y,z) = x^2y^4 \ \ {}^{g}\!/_{cm^3}$. Find the mass of the cube.

$$\int_1^3\int_0^2\int_{-1}^1 x^2y^4\ dx\,dy\,dz = \frac{128}{15}\ \text{grams}$$



 $-1 \le x \le 1,\ 0 \le y \le 2,\ 1 \le z \le 3$

3D Integrals:

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R_{uvw}} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|V_{xyz}(u,v,w)\right|\,du\,dv\,dw$$

$$V_{xyz}(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In my opinion, a good way to think about $\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz$ is as a calculation of the mass of R_{xyz} where $f(x,y,z)$ is the density of the solid at any given point (x,y,z).

Example 7: A 3D-Change of Variables with an Integrand

Compute $\iiint\limits_{R_{xyz}} 9y\,dx\,dy\,dz$ where R_{xyz} is the parallelepiped that is between the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $y = -2x + 3$.

All from Example 4:

$u = z - 3x$
$v = y - x$
$w = y + 2x$

$0 \le u \le 2$
$0 \le v \le 4$
$0 \le w \le 3$

$x = \dfrac{w - v}{3}$
$y = \dfrac{2v + w}{3}$
$z = u + w - v$

$\left|V_{xyz}(u,v,w)\right| = \dfrac{1}{3}$

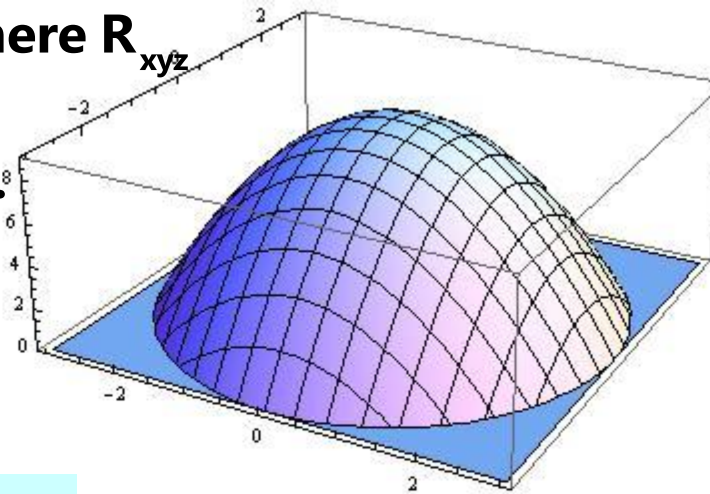
$$\iiint\limits_{R_{xyz}} 9y\,dx\,dy\,dz = \int_0^3\int_0^4\int_0^2 9y(u,v,w)\left|V_{xyz}(u,v,w)\right|\,du\,dv\,dw$$

$$= \int_0^3\int_0^4\int_0^2 9\left(\frac{2v + w}{3}\right)\frac{1}{3}\,du\,dv\,dw$$

$$= \int_0^3\int_0^4\int_0^2 (2v + w)\,du\,dv\,dw = 132$$

Example 8: A Traditional Triple Integral

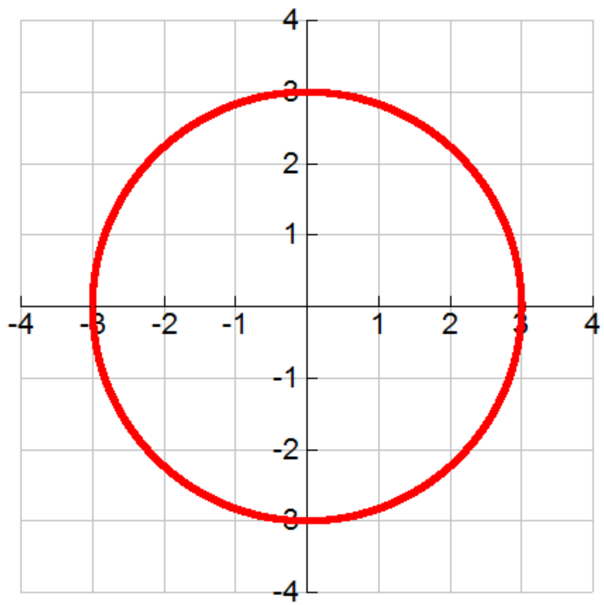
Set up, but do not compute $\iiint\limits_{R_{xyz}} 2y\,dx\,dy\,dz$ where R_{xyz} is the paraboloid $z = 9 - x^2 - y^2$ such that $z > 0$.



$$\iiint\limits_{R_{xyz}} 2y\,dz\,dy\,dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} 2y\,dz\,dy\,dx$$

z-top is the paraboloid $z = 9 - x^2 - y^2$ and z-bottom is the plane $z = 0$.

Find the intersection between $z = 9 - x^2 - y^2$ and $z = 0$: We get the circle $x^2 + y^2 = 9$, so our y-top is $y = \sqrt{9 - x^2}$ and our y-bottom is $y = -\sqrt{9 - x^2}$.



Finally, integrate x from $x = -3$ to $x = 3$.