Lesson 10 Transforming 3D Integrals

Example 1: Triple Integrals to Compute Volume

Created by Christopher Grattoni. All rights reserv $-1 \leq x \leq 1$, $0 \leq y \leq 2$, $1 \leq z \leq 3$ **I FIRM I REALLY A RECALL THE RECALL THAT A RECALL THAT A RECALL THE RECALL THAT A RECALL THE AVEC STATE OF A RECALL THAT A REPORT OF A RECALL THAT A RECALL THAT A RECALL THAT A REPORT OF A REPORT OF A REPORT OF A REPORT O Recall that in previous chapters we could find the length of an interval I R It follows that we can compute the volume of a 3-dimensional region R by calculating 1 dV. For this cube, the calculation is not exciting: 1** $\int_{1}^{3} \int_{-1}^{2} 1 \, dx \, dy \, dz = \int_{1}^{3} \int_{0}^{2} \left[x \right]_{x=1}^{x=1} dy \, dz$
 $= \int_{1}^{3} \left[2y \right]_{y=0}^{y=2} dz$
 $= \left[4z \right]_{z=1}^{z=3}$
 1 Created by Christopher Grattoni. All rights reserved $-1 \le x \le 1, 0 \le y \le 2, 1 \le z \le 3$ **1 0 1 1 dx dy dz** Ξ ∫∫∫ **3 2** $\mathsf{x} \!=\! \mathsf{1}$ $\mathsf{x}\!=\!-\mathsf{1}$ **1 0** $\mathbf{x}\right]_{\mathbf{x}=-1}^{\mathbf{x}=1}$ dy dz Ξ $=\int\!\!\int\!\!\left[\mathbf{x}\right]$ **3 y 2 y 0 1** 2y $\int_{\mathsf{y}=0}^{\mathsf{y}=2} \mathsf{dz}$ Ξ $=\int [2y]$ **z 3 z 1** $4z$ ^{$7z$ =} ═ ÷ $=\left[\!\left[\textbf{4z}\right]\!\right]$ **8**

Example 2: Triple Integrals to Compute Volume **Wolume**
We can spice things up a bit. We can find the volume of the region

above the xy-plane, the xz-plane, the yz-plane, and below the plane We can spice things up a bit.
above the xy-plane, the xz-p
given by x + y + 4z = 8 : <u>8 - x - y</u> $\frac{-x-y}{\sqrt{2}}$. So given by $x + y + 4z = 8$:

From by
$$
x + y + 4z = 8
$$
:

\nIf $x + y + 4z = 8$, then $z = \frac{8 - x - y}{4}$. So z -top is the plane $z = \frac{8 - x - y}{4}$ and z -bottom is the plane $z = 0$.

Find the intersection between z = _________ and $\mathbf{z} = \mathbf{0}$: We get the line $\mathbf{y} = \mathbf{8} - \mathbf{x}$, which is our **8 x y z 4** <u> Тит</u> ≡

y-top. Our y-bottom is the line **y** = **0**.

Finally, integrate x from $x = 0$ to $x = 8$.

8 8 x x y 0 8 0 0 4 1 dz dy dx $\int\int\int$

Example 2: Triple Integrals to Compute Volume **We can spice things up a bit. We can find the volume of the region**

above the xy-plane, the xz-plane, the yz-plane, and below the plane given by $x + y + 4z = 8$: We can spice things up a bit.
above the xy-plane, the xz-pl
given by x + y + 4z = 8 :

Example 3: Changing the Order of Integration

Entegration
Repeat Example 2 by computing \iint **dy dx dz where R is the region A**
Repeat Example 2 by computing ∭ dy dx dz where R is the region
above the xy-plane, the xz-plane, the yz-plane, and below the plane ∭ **R** above the xy-plane, the xz-plane, the yz-plane, and below the plane given by $x + y + 4z = 8$ If $\mathbf{x} + \mathbf{y} + 4\mathbf{z} = 8$, then $\mathbf{y} = \mathbf{8} - \mathbf{x} - 4\mathbf{z}$. So y-top $\mathbf{s} \cdot \mathbf{s} - \mathbf{x} - 4\mathbf{z}$ and y-bottom is the plane $\mathbf{y} = \mathbf{0}.$ 1.5 21.0 **x** + **y** + 4z = 8 and **y** = 0 intersect at the line **(our x-top). Our x-bottom is the**

x⁵

z

3

 $\mathop{\sf line}\nolimits {\mathbf x} = {\mathbf 0}.$

= о —

Finally, integrate z from z = to z 0 = 2.

Example 4: 3D-Change of Variables

<u>EXAIIIDIC 4. SD-CHANGE OF VALIADIES</u>
 Compute the volume of the parallelepiped described by the region R_{xyz}

**Compute the volume of the parallelepiped described by the region R_{xyz}
between the planes z = 3x and z = 3x + 2, y = x and y = x + 4, and y = -2x** Compute the vo
between the pla
and y = -2x + <mark>3</mark>. of the parallelepiped described by the region R_{xyz}
= 3x and z = 3x + 2, y = x and y = x + 4, and y = $-2x$ ute the volume α
en the planes z =
= -2x + 3.

u z 3x, and let u run from 0 to 2 v = **y** – **x**, and let **v** run from 0 to 4 **w y 2x, and let w run from 0 to 3**

The Volume Conversion Factor:

$$
\iiint\limits_{R_{xyz}} 1 dx dy dz = \iiint\limits_{R_{uvw}} |V_{xyz}(u, v, w)| du dv dw
$$

Let T(u,v,w) be a transformation from uvw-space to xyz-space.

1 2 3 Extract T(u, v, w) be a transformation from uvw-space to xyz-space.
 1 2 That is, T(u,v,w) = (T₁(u, v, w), T₂(u, v, w), T₃(u, v, w)) = (x(u, v, w), y(u, v, w), z(u, v, w)). $= (T, (u, v, w), T, (u, v, w), T, (u, v, w)) =$

That is, T(u,v,w) = (T₁(u,v,w), T₂(u,v,w), T₃(u,v,w)) = (x(u,v,w), y(u,v,w), z(u,v,w)).
\nThen V_{xyz}(u,v,w) =
$$
\begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & \frac{\partial T_3}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & \frac{\partial T_3}{\partial v} \\ \frac{\partial T_1}{\partial w} & \frac{\partial T_2}{\partial w} & \frac{\partial T_3}{\partial w} \end{vmatrix}
$$
 or V_{xyz}(u,v,w) = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$
\nNote: Just like A_{xy}(u, v), we need V_{xyz}(u, v, w) to be positive. Hence, the absolute

value bars in the formula above. *ow ow ow Sigms are:*
 value bars in the formula above.
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= 3x and z = 3x + 2, y = x and y = x + 4, and y = $-2x$ ute the volume **c**
en the planes z =
= -2x + 3. $\mathbf{w} - \mathbf{v}$ **0 0 1**

 $V_{\text{avg}}(u, v, w) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & w \end{vmatrix}$

Ξ

3 3

3 3

3 3 ¹

┯

1

3

1

3

 $=$ $-$

Use $\bar{-}$.

1 1 ¹

1 2

1 1

3 3

$$
x = \frac{1}{3}
$$

\n
$$
u = z - 3x
$$

\n
$$
v = y - x
$$

\n
$$
y = \frac{2v + w}{3}
$$

\n
$$
w = y + 2x
$$

\n
$$
z = u + w - v
$$

\n
$$
y = \frac{2v + w}{3}
$$

$$
\int_{0}^{3}\int_{0}^{4}\int_{0}^{2}|V_{xyz}(u,v,w)|du dv dw = \int_{0}^{3}\int_{0}^{4}\int_{0}^{2}\frac{1}{3}du dv dw
$$
\n
$$
= 8
$$

Example 4: 3D-Change of Variables

<u>EXAIIIDIC 4. SD-CHANGE OF VALIADIES</u>
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and y = -2x + <mark>3</mark>. of the parallelepiped described by the region R_{xyz}
= 3x and z = 3x + 2, y = x and y = x + 4, and y = $-2x$ ute the volume **c**
en the planes z =
= –2x + 3. **Check : No need to study this, it's just nice to verify this works...**

Check :
For a parallelepiped ₍
V_{parallelepiped} = (X × Y)•Z **For a parallelepiped generated by three intersecting vectors X, Y and Z,** lelepiped gene
= (X × Y)•Z

 $\bf{V}_{\rm parallelepiped} = (X \times Y) \bullet Z$
Parallelepiped is generated by vectors (1,1,3), $\left(-\frac{4}{3}, \frac{8}{3}, -4\right)$, and (0,0,2). $\frac{4}{3}$, $\frac{8}{3}$ $\left(-\frac{4}{3}, \frac{8}{3}, -4\right)$, and (0,0) $\left(-\frac{4}{3}, \frac{8}{3}, -4\right)$, and (0,0)

$$
V_{\text{parallellepiped}} = \left((1,1,3) \times \left(-\frac{4}{3}, \frac{8}{3}, -4 \right) \right) \cdot (0,0,2) = 8
$$

Example 5: A Mathematica-Assisted Change of Variables

 $\mathbf{(4-s^2)}\mathbf{(cos(t),sin(t),0)}$ **Find the volume of the "football" whose outer skin is described by the** Find the volume of the "football" whose outer skin is described by
parametric equation F(s, t) = (0, 0, 4s) + $\left(4-s^2\right) \left(\cos(t),\sin(t),0\right)$ for nd t
aran
2 ≤ s **F(s 2 volume of**
Pric equatio
2 and 0 ≤ t **t** $\overbrace{ }^{11}$ (football" whose outer skin
, t) = (0, 0, 4s) + $\left(4 - s^2 \right)$ (cos(1 Find the volume of the rootball
parametric equation F(s, t) = (0, 0
-2 \leq s \leq 2 and 0 \leq t \leq 2 π . =

2 . First, fill in the football:

 ${\bf F}({\bf r},{\bf s},{\bf t})=({\bf 0},{\bf 0},{\bf 4}{\bf s})+{\bf r}\Big({\bf 4}-{\bf s}^2\Big) \Big({\bf cos(t),sin(t),0}\Big)$

 $-2 \leq$ **s** \leq **2** $\mathbf{0} \leq \mathbf{t} \leq 2\pi$

0 r 1

Change of variables:

$$
x(r,s,t) = r(4-s2)cos(t)
$$

$$
y(r,s,t) = r(4-s2)sin(t)
$$

$$
z(r,s,t) = 4s
$$

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0 2 0

π

π

Example 5: A Mathematica-Assisted Change of Variables

Find the volume of the "football" whose outer skin is described **b**
parametric equation F(s, t) = (0, 0, 4s) + (4 - s²)(cos(t), sin(t), 0) for **ootball" whose outer skin
) = (0, 0, 4s) + (4 – s²) (cos(1 Find the volume of the "football" whose outer skin is described by the** ind the volume of
arametric equatic
2 ≤ s ≤ 2 and 0 ≤ t **1e** '
F(s,
2π. Find the volume of the "football
parametric equation F(s, t) = (0, 0
-2 \leq s \leq 2 and 0 \leq t \leq 2 π . $^{2})\big(\cos(t),\sin(t),0\big)$ **(** ∂ x ∂ y ∂ z \vert **x** ∂ **y** ∂ **z** $\frac{1}{r}$ $\frac{1}{r}$ $\frac{1}{r}$ $\overline{\partial} \mathbf{r}$ $\overline{\partial} \mathbf{r}$ $\overline{\partial} \mathbf{r}$ **x** ∂ **y** ∂ **z** $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$ **V (r,s,t)** $=\left|\frac{\partial x}{\partial s} \quad \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial s}\right|$ $\overline{\mathbf{s}}$ $\overline{\partial}$ $\overline{\mathbf{s}}$ $\overline{\partial}$ $\overline{\partial}$ **xyz x z ^y** $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$ $\overline{\partial t}$ $\overline{\partial t}$ $\overline{\partial t}$ $\frac{1}{\partial t}$ $\frac{1}{\partial t}$ z $= -4(16r - 8rs^2 + rs^4)$ (Mathematica)

$$
\int_{0}^{2\pi/2}\int_{0}^{1}|V_{xyz}(r,s,t)|\,dr\,ds\,dt=\frac{2048}{15}\pi\quad\text{(Mathematica)}
$$

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Example 5: A Mathematica-Assisted Change of Variables

 $^{2})\big(\cos(t),\sin(t),0\big)$ **Find the volume of the "football" whose outer skin is described by the** Find the volume of the "football" whose outer skin is described **b**
parametric equation F(s, t) = (0, 0, 4s) + (4 - s²)(cos(t), sin(t), 0) for ind the volume of
arametric equatic
2 ≤ s ≤ 2 and 0 ≤ <mark>t</mark> **(1e** '
F(s,
2π. **ootball" whose outer skin
) = (0, 0, 4s) + (4 – s²) (cos(1** Find the volume of the "football
parametric equation F(s, t) = (0, 0
-2 \leq s \leq 2 and 0 \leq t \leq 2 π .

Check using solids of revolution:

$$
\pi \int_{-8}^{8} \left(4 - \left(\frac{x}{4}\right)^2\right)^2 dx = \frac{2048}{15} \pi
$$

Example 6: Beyond Volume **Calculations**

It is easy to see that $\iint dV$ **computes the volume of a solid. But it's harder to interpret for fixes)**
harder to interpret for f(x,y,z) dV since f(x,y,z) lives in 4 dimensions. **R R We need an example to keep referring back to to give us some intuition. A cube of varying density has its density at** each point (x,y,z) described by
f(x,y,z) = x²y⁴ (2)_{cm³}. Find the mass **each point (x,y,z) described by** 3.0 **2 4 ^g of the cube.**

$$
\int_{1}^{3} \int_{0}^{2} \int_{-1}^{1} x^2 y^4 dx dy dz = \frac{128}{15}
$$
 grams

3D Integrals:

**xyz uvw xyz <u>BD Integrals:</u>
** $\iint_{R_{xyz}} f(x,y,z) dx dy dz = \iiint_{R_{uvw}} f(x(u,v,w),y(u,v,w),z(u,v,w)) |V_{xyz}(u,v,w)| du dv dw$ **xyz x** *o*y *o*z **u x** *c*y *c*z **w V (u,v, u u** w) = $\frac{\partial x}{\partial x}$ $\frac{\partial y}{\partial y}$ **w v w z v** *c***v** *c***v** $\partial {\bf x} = \partial$ ∂ x ∂ v ∂ $\partial \mathbf{u} = \partial$ $\partial {\bf x}$ $\partial {\bf v}$ ∂ $\partial \mathbf{w}$ ∂ $\partial {\bf v}$ $\partial {\bf v}$ ∂ 6 д 6 Ξ

xyz R In my opinion, a good way to think about $\int_0^{\infty} f(x,y,z) dx dy dz$ **is** In my opinion, a good way to think about $\iiint\limits_{R_{xyz}} f(x,y,z) dx dy dz$ is
as a calculation of the mass of R_{xyz} where $f(x,y,z)$ is the density

of the solid at any given point (x,y,z). as a calculation of the mass of R_{xyz} where $f(x, y, z)$ is the density

Example 7: A 3D-Change of Variables with an Integrand

xyz R x with an Integrand
 Compute \iint 9y dx dy dz where R_{xyz} is the parallelepiped that is between **Compute** $\iiint_{R_{xyz}} 9y dx dy dz$ **where R_{xyz} is the parallelepiped that is betwe**
the planes $z = 3x$ and $z = 3x + 2$, $y = x$ and $y = x + 4$, and $y = -2x$ and $\int\!\!\!\int\!\!\!\int$

the planes $z = 3x$ a
y = $-2x + 3$. $\mathbf{w} - \mathbf{v}$

All from Example 4:

$$
x = \frac{3}{3}
$$

u = z - 3x 0 \le u \le 2
v = y - x 0 \le v \le 4 y = \frac{2v + w}{3}
w = y + 2x 0 \le w \le 3 z = u + w - v $|V_{xyz}(u, v, w)| = \frac{1}{3}$

$$
\iiint_{R_{xyz}} 9y \, dx \, dy \, dz = \int_{0}^{3} \int_{0}^{4} \int_{0}^{2} 9y(u, v, w) |V_{xyz}(u, v, w)| \, du \, dv \, dw
$$

=
$$
\int_{0}^{3} \int_{0}^{4} \int_{0}^{2} 9\left(\frac{2v + w}{3}\right) \frac{1}{3} \, du \, dv \, dw
$$

=
$$
\int_{0}^{3} \int_{0}^{4} \int_{0}^{2} (2v + w) \, du \, dv \, dw
$$

Example 8: A Traditional Triple Integral

xyz x
 Set up, but do not compute ∭2y dx dy dz where R_{×yg} R R_{xyz} ² – V^2 **is the paraboloid** $z = 9 - x^2 - y^2$ **such that** $z > 0$ **.**

$$
\iiint\limits_{R_{xyz}} 2y \, dz \, dy \, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} 2y \, dz \, dy \, dx
$$

z-top is the paraboloid $\mathbf{z} = \mathbf{9} - \mathbf{x}^2 - \mathbf{y}^2$ **and z-bottom is the plane ^z 0.**

Find the intersection between $\mathbf{z} = \mathbf{9} - \mathbf{x}^2 - \mathbf{y}^2$ and $\mathbf{x} = \sqrt{9-x^2}$ and our y-bottom is $\mathbf{y} = -\sqrt{9-x^2}$. $\mathbf{z} = \mathbf{0}$: We get the circle $\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{9}$, so our y-top

Finally, integrate x from $x = -3$ **to** $x = 3$ **.**

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