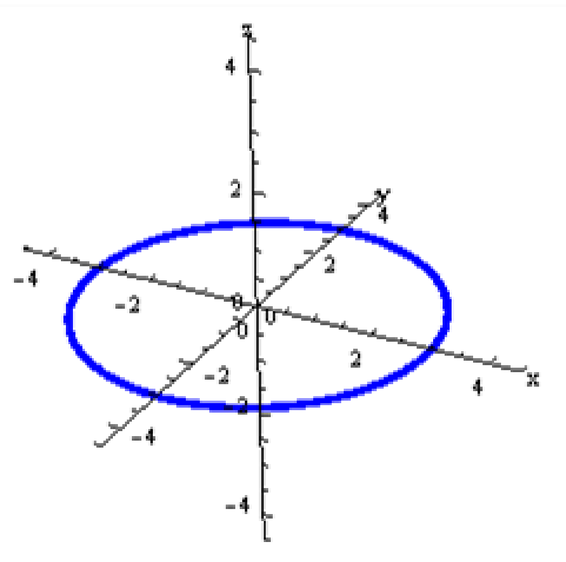
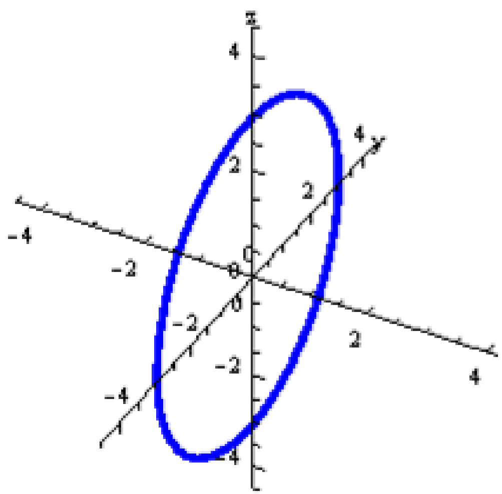
Lesson 11

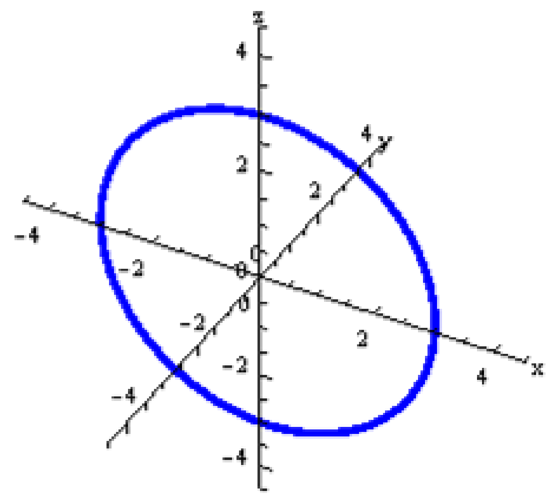
Spherical and Cylindrical Coordinates

Parameterizing a Sphere

To parameterize a sphere of radius r centered at the origin, we know that the cross section of the sphere with each of the xy-, xz-, and yz-planes should be a circle of radius r. We can use this to build our parameterization:



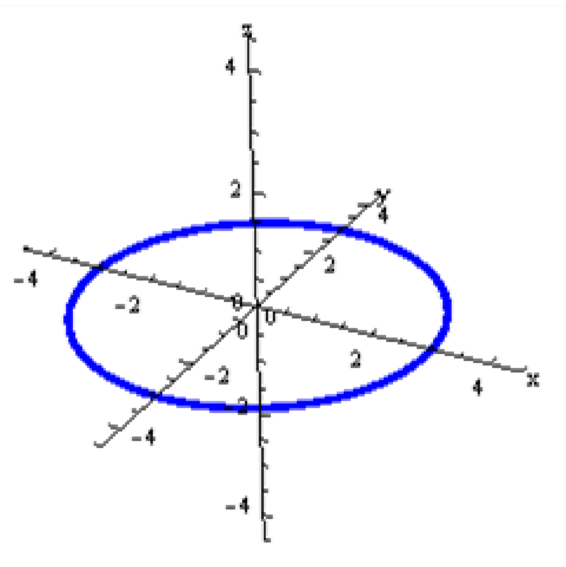


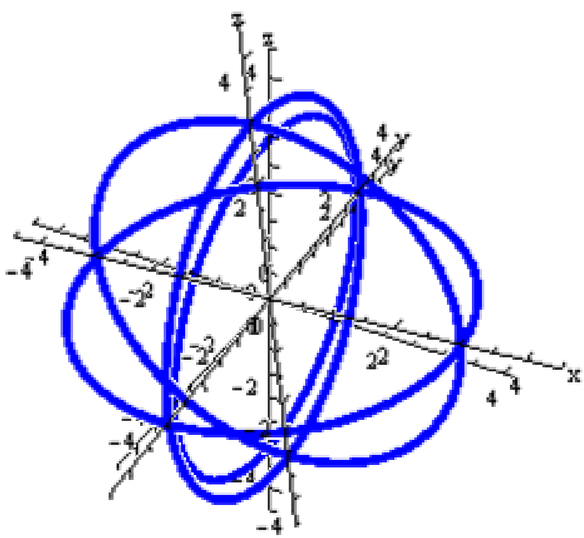


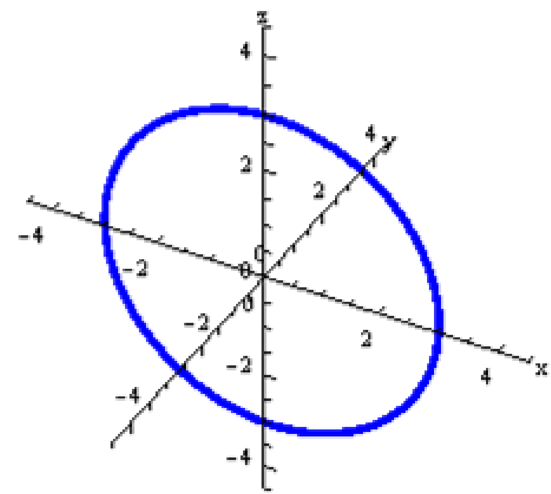
$\left(r\cos(t), r\sin(t), 0\right)$ $\left(0, r\sin(s), r\cos(s)\right)$ $\left(r\sin(s), 0, r\cos(s)\right)$

Parameterizing a Sphere

Put it all together and what do you get…



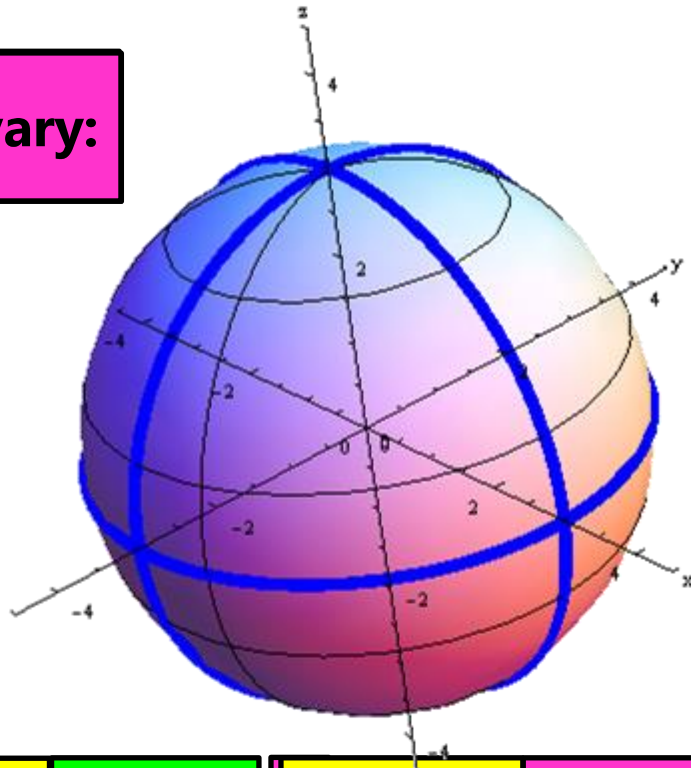


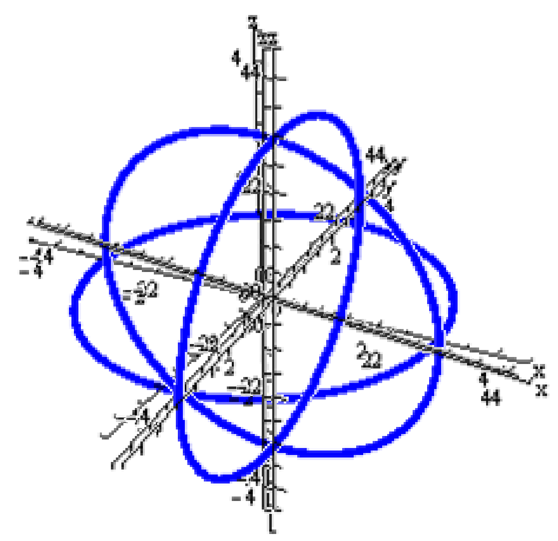


$\left(r\cos(t), r\sin(t), 0\right)$ $\left(r\cos(t)\sin(s), r\sin(t)\sin(s), r\cos(s)\right)$ $\left(r\sin(s), 0, r\cos(s)\right)$

Parameterizing a Sphere

Let $t = \frac{\pi}{2}$ and let s vary:





$$\left(r\cos(t)\sin(s),\ r\sin(t)\sin(s),\ r\cos(s)\right)$$

$$0 \le t \le 2\pi,\ 0 \le s \le \pi$$

Spherical Coordinates

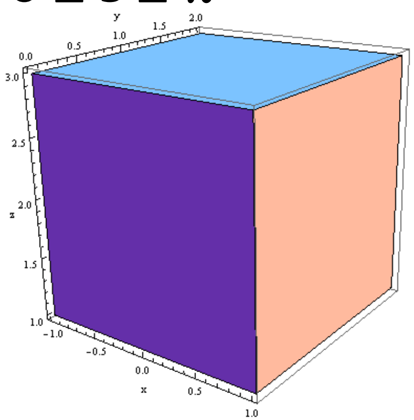
This parameterization is a map from spherical coordinates, rst-space, to rectangular coordinates, xyz-space:

rst-Rectangular Prism

$0 \leq r \leq r_0$

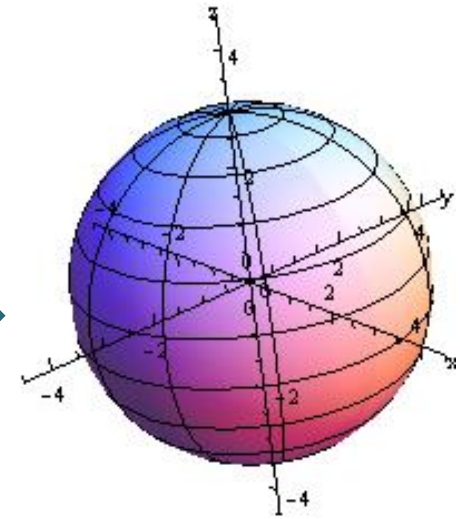
$0 \leq t \leq 2\pi$

$0 \leq s \leq \pi$



xyz-Sphere

radius: r_0



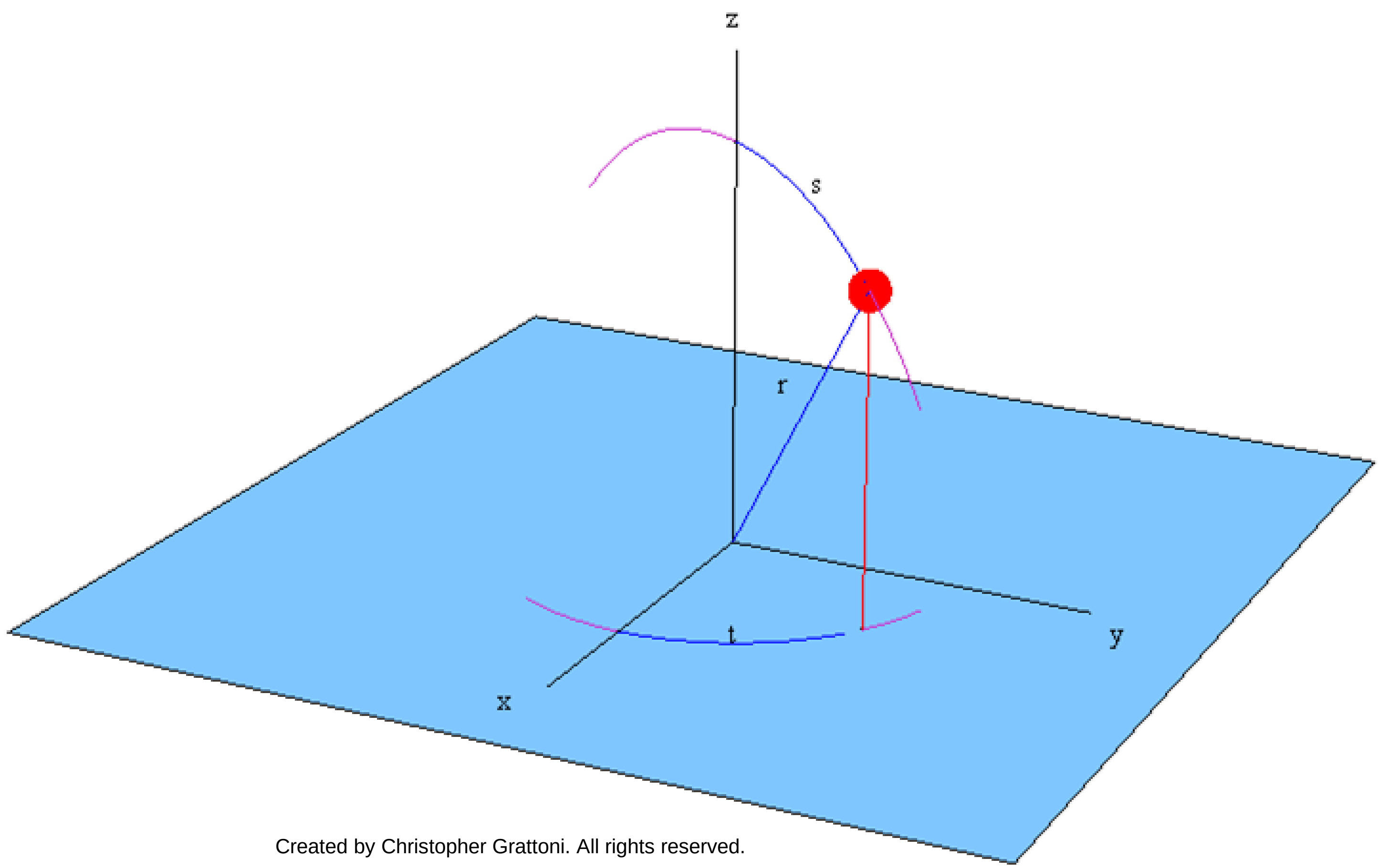
$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$

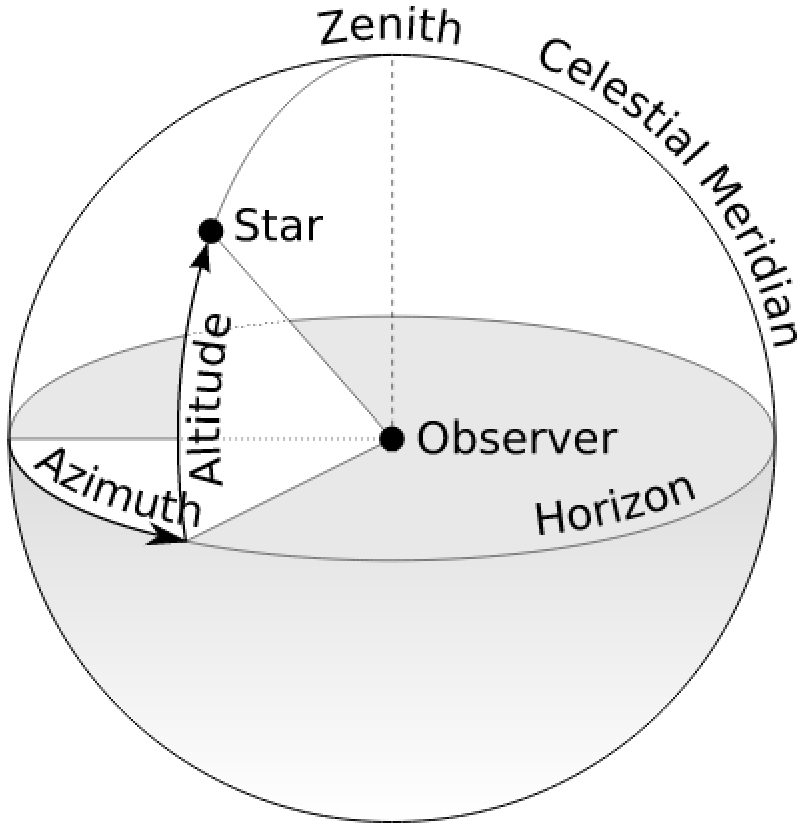
Spherical Coordinates

Spherical coordinates are ordered triples (r,s,t) with r as the radius, s as the zenith angle, and t as the azimuthal angle.



Spherical Coordinates and Ancient Astronomy

- Zenith: Arabic for “over one’s head”, to Old Spanish, to Medieval Latin, to Middle French, to Middle English, to Modern English
- Azimuth: Arabic for “the way”, Medieval Latin, to Middle English, to Modern English



The Volume Conversion Factor

Hence, we can find a volume conversion factor for our mapping:

$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$

$$V_{xyz}(r,s,t) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(t)\sin(s) & \sin(t)\sin(s) & \cos(s) \\ r\cos(t)\cos(s) & r\sin(t)\cos(s) & -r\sin(s) \\ -r\sin(t)\sin(s) & r\cos(t)\sin(s) & 0 \end{vmatrix}$$

$$= r^2\sin(s)$$

Summary: Spherical Coordinates

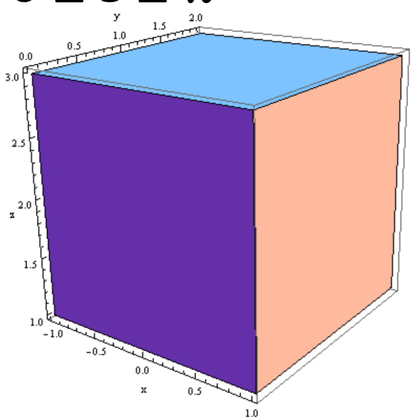
This parameterization is a map from spherical coordinates, rst-space, to rectangular coordinates, xyz-space:

rst-Rectangular Prism

$0 \le r \le r_0$

$0 \le t \le 2\pi$

$0 \le s \le \pi$



$$V_{xyz}(r,s,t) = r^2 \sin(s)$$

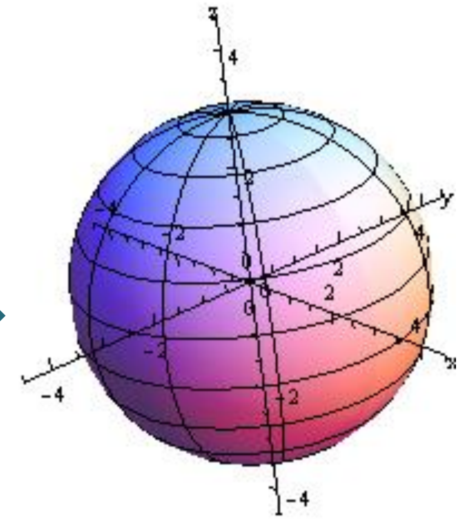
$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$

xyz-Sphere

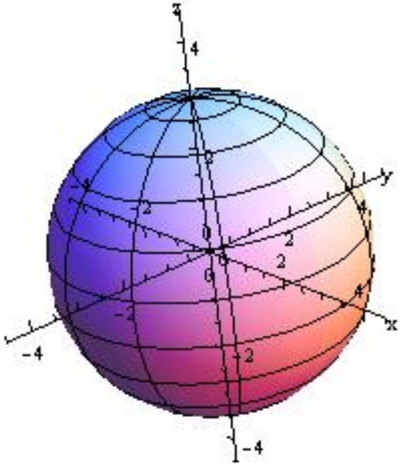
radius: r_0



Example 1: Find the Volume of a Sphere

Find the volume of the solid described by $x^2 + y^2 + z^2 \leq 9$ using a triple integral:

Unpleasant xyz-Space Integral: $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} dz\,dy\,dx$



Nicer rst-Space Integral: $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{3}\left|r^2\sin(s)\right| dr\,ds\,dt$

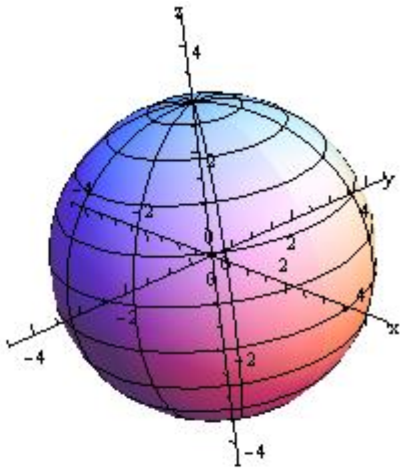
$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$

Example 1: Find the Volume of a Sphere

Find the volume of the solid described by $x^2 + y^2 + z^2 = 9$ using a triple integral:



$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \left| r^2 \sin(s) \right| dr\, ds\, dt = \int_0^{2\pi}\int_0^{\pi} \left[\frac{r^3}{3} \sin(s) \right]_{r=0}^{r=3} ds\, dt$$

$$= \int_0^{2\pi}\int_0^{\pi} 9\sin(s)\, ds\, dt$$

$$= \int_0^{2\pi} \left[-9\cos(s) \right]_{s=0}^{s=\pi} dt$$

$$= \int_0^{2\pi} -9\left(\cos(\pi) - \cos(0)\right) dt$$

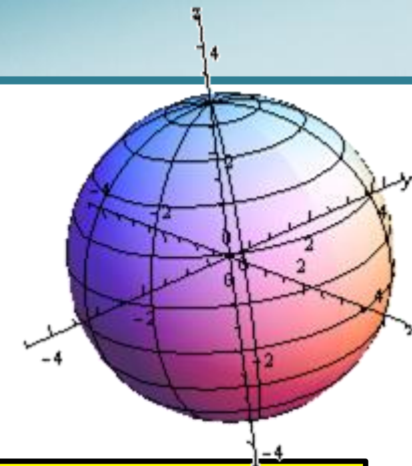
$$= \int_0^{2\pi} 18\, dt = 36\pi$$

Example 2: Add an Integrand!

Find $\iiint\limits_R x^2+y^2+z^2\,dx\,dy\,dz$ over the region contained

within $x^2+y^2+z^2=9$:

$$x(r,s,t)=r\cos(t)\sin(s)$$
$$y(r,s,t)=r\sin(t)\sin(s)$$
$$z(r,s,t)=r\cos(s)$$



$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\left(\left(r\cos(t)\sin(s)\right)^2+\left(r\sin(t)\sin(s)\right)^2+\left(r\cos(s)\right)^2\right)\left(r^2\sin(s)\right)dr\,ds\,dt$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\left(r^2\sin^2(s)\left(\cos^2(t)+\sin^2(t)\right)+\left(r\cos(s)\right)^2\right)\left(r^2\sin(s)\right)dr\,ds\,dt$$

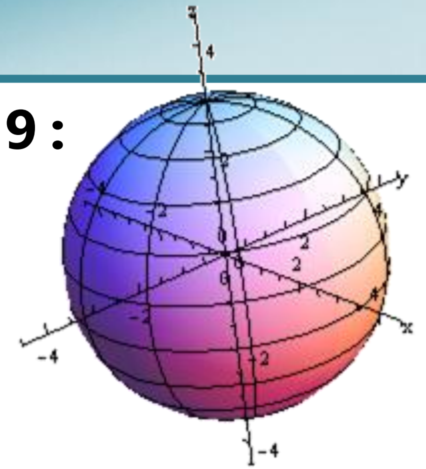
$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\left(r^2\sin^2(s)+r^2\cos^2(s)\right)\left(r^2\sin(s)\right)dr\,ds\,dt$$

Shortcut??

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{3}r^4\sin(s)\,dr\,ds\,dt$$

Example 2: Add an Integrand!

Find $\iiint\limits_R x^2+y^2+z^2\,dx\,dy\,dz$ **over the region** $x^2+y^2+z^2=9$**:**



$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} r^4\sin(s)\,dr\,ds\,dt=\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^5}{5}\sin(s)\right]_{r=0}^{r=3}ds\,dt$$

$$=\frac{243}{5}\int_0^{2\pi}\int_0^{\pi}\sin(s)\,ds\,dt$$

$$=\frac{243}{5}\int_0^{2\pi}\left[-\cos(s)\right]_0^{\pi}dt$$

$$=\frac{243}{5}\int_0^{2\pi}2\,dt$$

$$=\frac{972}{5}\pi$$

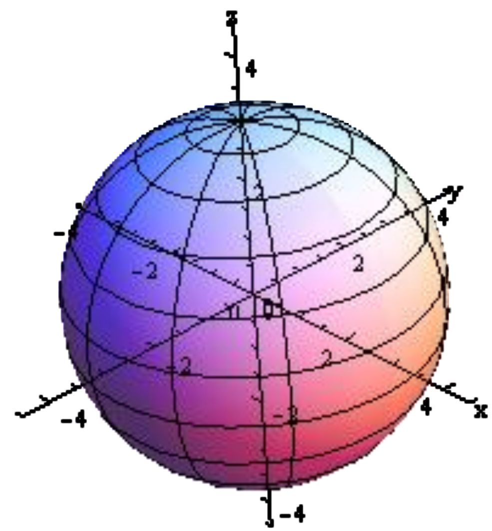
Example 3: Playing with Parameterizations

Fix $r = 3$ and let $0 \leq t \leq 2\pi$. Plot the results of letting s vary from 0 to π:

$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$



Try yourself in Mathematica!

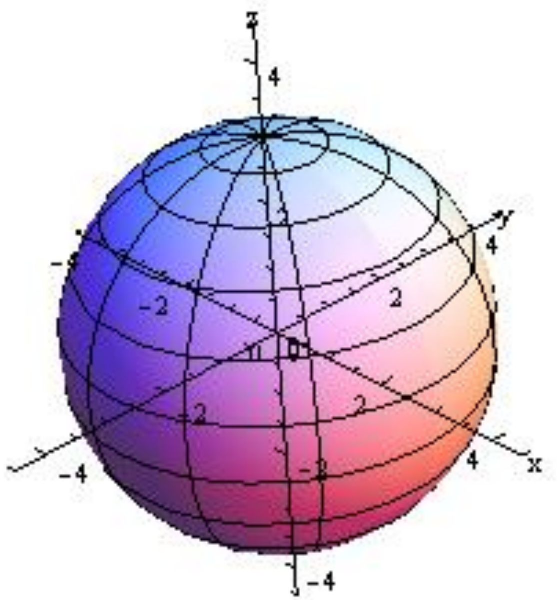
Example 4: Playing with Parameterizations Again

Fix $r = 3$ and let $0 \le s \le \pi$. Plot the results of letting t vary from 0 to 2π:

$$x(r,s,t) = r\cos(t)\sin(s)$$

$$y(r,s,t) = r\sin(t)\sin(s)$$

$$z(r,s,t) = r\cos(s)$$



Try yourself in Mathematica!

Example 5: Playing with Parameterizations (Part 3)

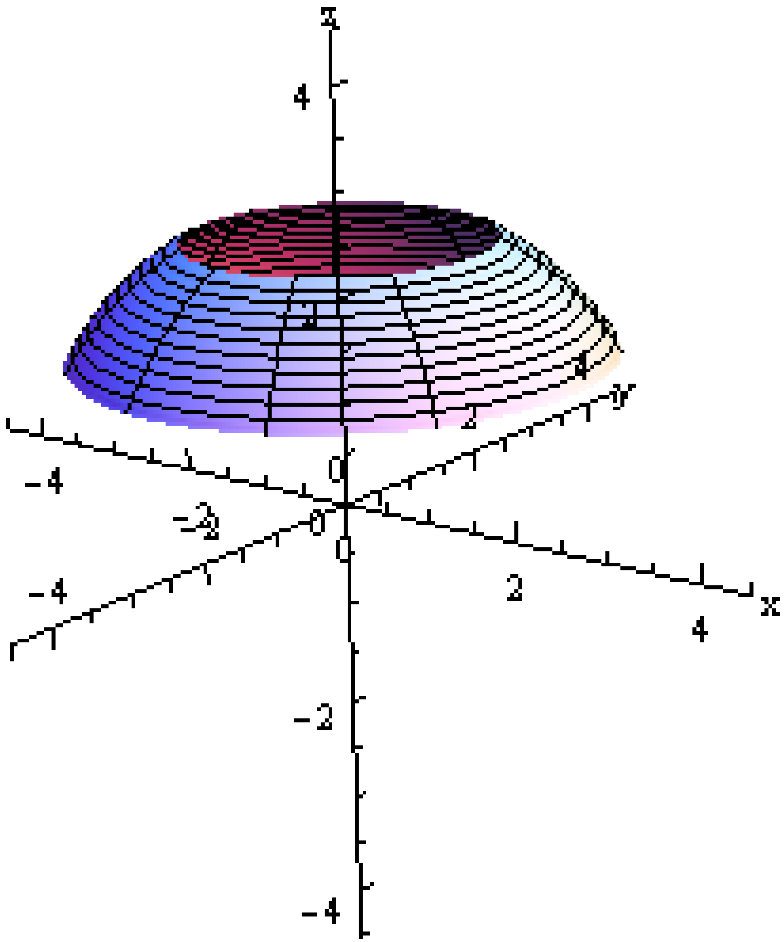
Find values for r, s, and t that could give the following plot:

Possible Answer :

$$r = 3$$

$$\frac{\pi}{6} \le s \le \frac{\pi}{4}$$

$$0 \le t \le 2\pi$$



Example 6: Playing with Parameterizations (Part 4)

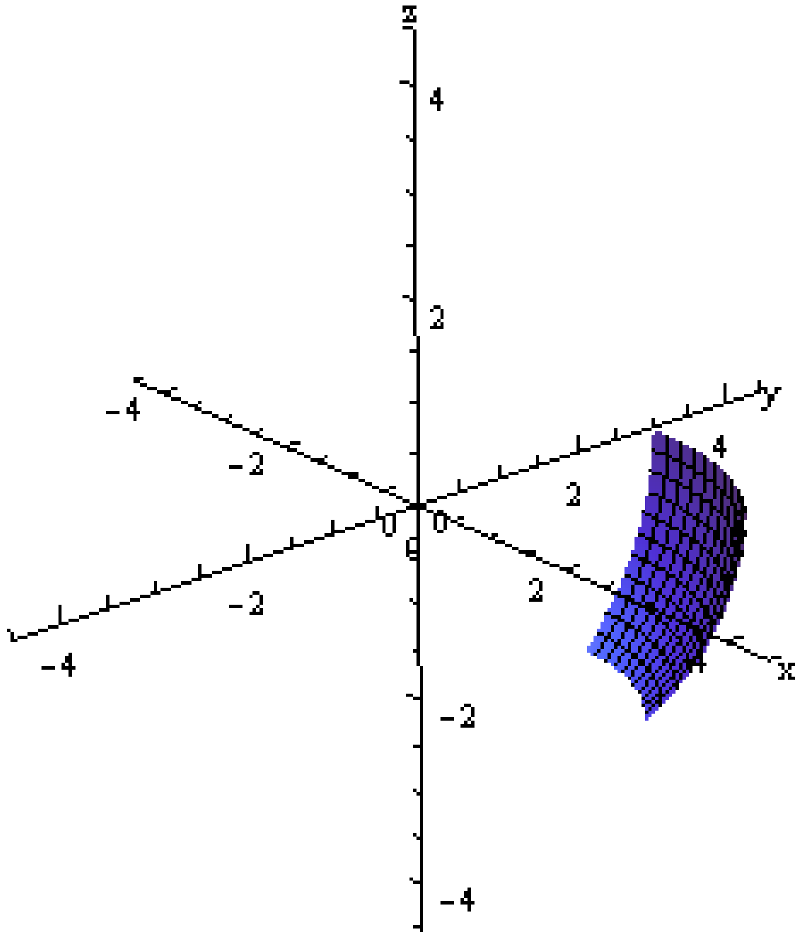
Find values for r, s, and t that could give the following plot:

Possible Answer :

$$r = 3$$

$$\frac{\pi}{2} \le s \le \frac{3\pi}{4}$$

$$\frac{\pi}{4} \le t \le \frac{\pi}{2}$$



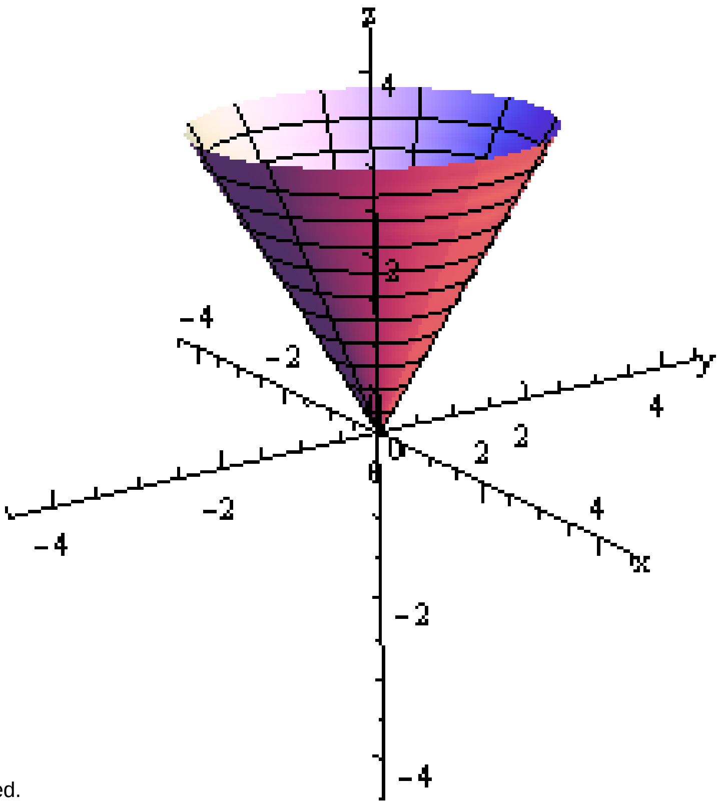
Example 7: Cones

Plot out a cone with a slant height of 4 whose slant height and altitude form an angle of $\frac{\pi}{6}$ radians.

$$x(r,s,t) = r\cos(t)\sin(s)$$
$$y(r,s,t) = r\sin(t)\sin(s)$$
$$z(r,s,t) = r\cos(s)$$

Fix $s = \frac{\pi}{6}$ **and let** $0 \le t \le 2\pi,\ 0 \le r \le 4:$

$$x(r,s,t) = r\cos(t)\sin\left(\frac{\pi}{6}\right)$$
$$y(r,s,t) = r\sin(t)\sin\left(\frac{\pi}{6}\right)$$
$$z(r,s,t) = r\cos\left(\frac{\pi}{6}\right)$$



Cylindrical Coordinates

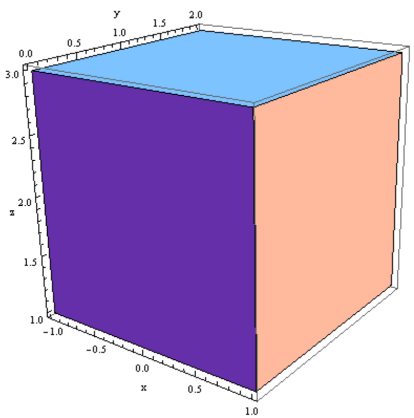
This parameterization is a map from cylindrical coordinates, rst-space, to rectangular coordinates, xyz-space:

rst-Rectangular Prism

$0 \leq r \leq r_0$

$0 \leq t \leq 2\pi$

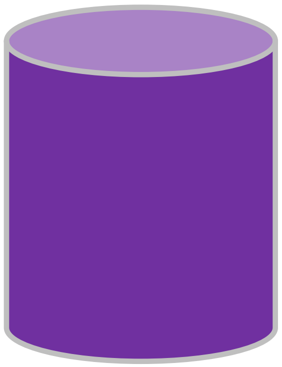
$0 \leq s \leq \pi$



xyz-Cylinder

radius: r_0

height : s_0



$$x(r,s,t) = r\cos(t)$$

$$y(r,s,t) = r\sin(t)$$

$$z(r,s,t) = s$$

The Volume Conversion Factor

We can find a volume conversion factor for our mapping:

$$x(r,s,t) = r\cos(t)$$
$$y(r,s,t) = r\sin(t)$$
$$z(r,s,t) = s$$

$$V_{xyz}(r,s,t) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \\ -r\sin(t) & r\cos(t) & 0 \end{vmatrix}$$

We will use $\left|V_{xyz}(r,s,t)\right| = r.$

Summary: Cylindrical Coordinates

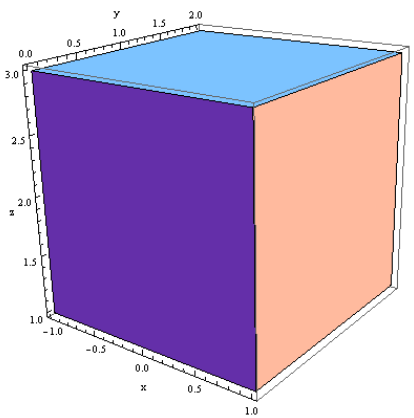
This parameterization is a map from cylindrical coordinates, rst-space, to rectangular coordinates, xyz-space:

rst-Rectangular Prism

$0 \le r \le r_0$

$0 \le t \le 2\pi$

$0 \le s \le \pi$



$$\left|V_{xyz}(r,s,t)\right| = r$$

$$x(r,s,t) = r\cos(t)$$

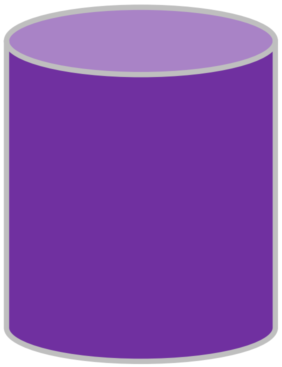
$$y(r,s,t) = r\sin(t)$$

$$z(r,s,t) = s$$

xyz-Cylinder

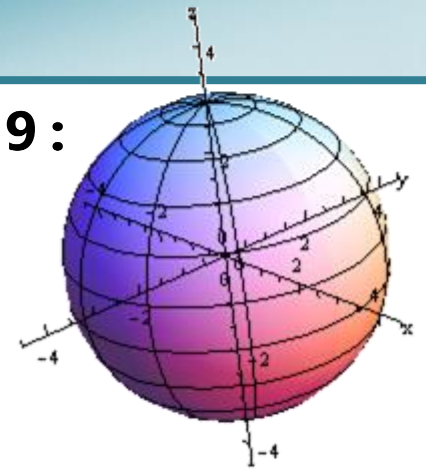
radius: r_0

height : s_0



Example 8: Again With Cylindrical Coordinates

Find $\iiint_R x^2 + y^2 + z^2 dx\,dy\,dz$ over the region $x^2 + y^2 + z^2 = 9$:



$x(r,s,t) = r\cos(t)$

$y(r,s,t) = r\sin(t)$

$z(r,s,t) = s$

Use $\left|V_{xyz}(r,s,t)\right| = r$

$x^2 + y^2 + z^2 = 9$

$(r\cos(t))^2 + (r\sin(t))^2 + s^2 = 9$

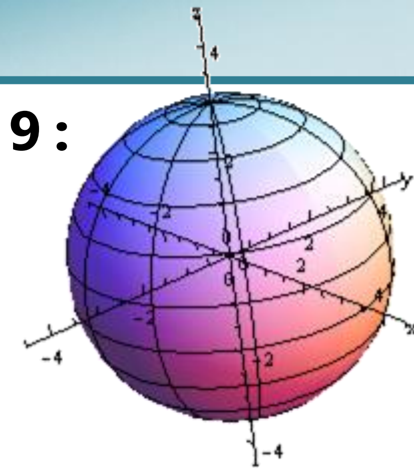
$r^2 + s^2 = 9$

$r = \sqrt{9 - s^2}$ with $-3 \le s \le 3$

$$\int_0^{2\pi}\int_{-3}^{3}\int_0^{\sqrt{9-s^2}} \left(r^2 + s^2\right)(r)\,dr\,ds\,dt = \frac{972}{5}\pi$$

Example 9: Now With Custom Coordinates

Find $\iiint\limits_R x^2+y^2+z^2 dx\,dy\,dz$ **over the region** $x^2+y^2+z^2=9$ **:**



$$x^2+y^2+z^2=9$$
$$(r\cos(t))+(r\sin(t))^2+s^2=9$$
$$r^2+s^2=9$$
$$r=\sqrt{9-s^2}\text{ with }-3\le s\le 3$$

$$x(r,s,t)=r\sqrt{9-s^2}\cos(t)$$
$$y(r,s,t)=r\sqrt{9-s^2}\sin(t)$$
$$z(r,s,t)=s$$
$$\text{Calculate } V_{xyz}(r,s,t)=r(s^2-9)$$

Mathematica

$$\int_0^{2\pi}\int_{-3}^{3}\int_0^{1}\left(x(r,s,t)^2+y(r,s,t)^2+z(r,s,t)^2\right)\left|r(s^2-9)\right|dr\,ds\,dt=\frac{972}{5}\pi$$

Make sure to use absolute value bars with this technique.

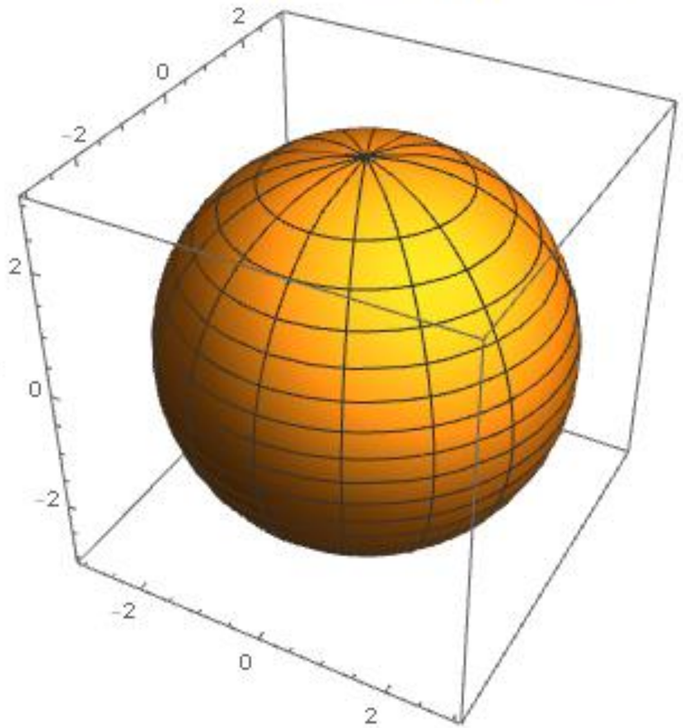
Example 9: Plot With Custom Coordinates

Plot $x^2 + y^2 + z^2 = 9$ using your custom coordinates :

$$x(r,s,t) = r\sqrt{9-s^2}\cos(t)$$
$$y(r,s,t) = r\sqrt{9-s^2}\sin(t)$$
$$z(r,s,t) = s$$

```
In[131]:= x[r_, s_, t_] = r Sqrt[9 - s^2] Cos[t];
          y[r_, s_, t_] = r Sqrt[9 - s^2] Sin[t];
          z[r_, s_, t_] = s;
          ParametricPlot3D[{x[1, s, t], y[1, s, t], z[1, s, t]}, {s, -3, 3}, {t, 0, 2 Pi}]
```

Out[134]=



Example 10: Plotting a "Peanut"

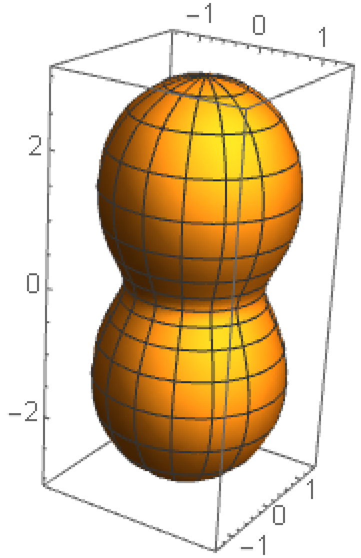
Here's a peanut plotted with spherical coordinates :

In[1]:=
```
Clear[x, y, z, r, s, t, rad]
rad[s_] = Cos[2 s] + 2;

x[r_, s_, t_] = r Cos[t] Sin[s];
y[r_, s_, t_] = r Sin[t] Sin[s];
z[r_, s_, t_] = r Cos[s];

ParametricPlot3D[{x[rad[s], s, t], y[rad[s], s, t], z[rad[s], s, t]}, {s, 0, Pi}, {t, 0, 2 Pi}]
```

Out[6]=



Example 10: Plotting a "Peanut"

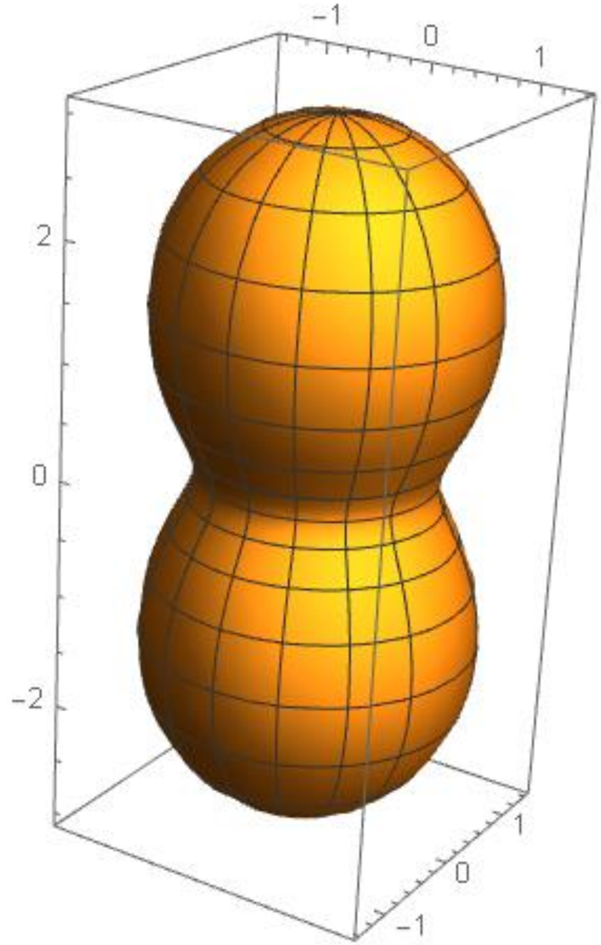
Here's its volume using spherical coordinates :

```
Clear[x, y, z, r, s, t, rad]
rad[s_] = Cos[2 s] + 2;

x[r_, s_, t_] = r Cos[t] Sin[s];
y[r_, s_, t_] = r Sin[t] Sin[s];
z[r_, s_, t_] = r Cos[s];
```

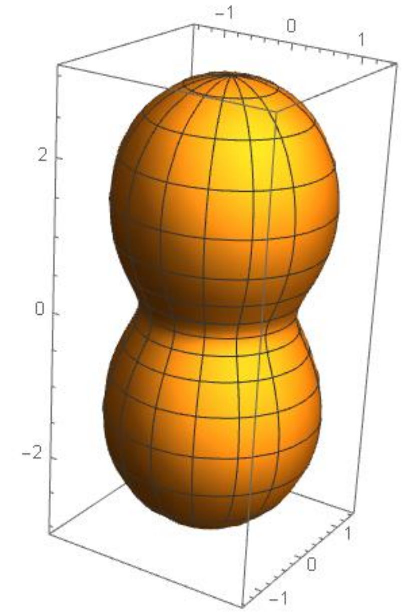
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{rad[s]} r^2\, Sin[s]\, dr\, ds\, dt$$

$$\frac{916\,\pi}{105}$$



Example 10: Plotting a "Peanut"

Here's its volume using custom coordinates :




```
Clear[x, y, z, r, s, t, rad]
rad[s_] = Cos[2 s] + 2;

x[r_, s_, t_] = r rad[s] Cos[t] Sin[s];
y[r_, s_, t_] = r rad[s] Sin[t] Sin[s];
z[r_, s_, t_] = r rad[s] Cos[s];


Clear[gradx, grady, gradz, Vxyz];
gradx[r_, s_, t_] = {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] = {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradz[r_, s_, t_] = {D[z[r, s, t], r], D[z[r, s, t], s], D[z[r, s, t], t]};

Vxyz[r_, s_, t_] = Simplify[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]
```

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \text{Abs}[\text{Vxyz}[r, s, t]]\, dr\, ds\, dt$$

$$r^2\ (2 + \text{Cos}[2\ s])^3\ \text{Sin}[s]$$

$$\frac{916\,\pi}{105}$$