Lesson 12:

Surface Integrals and the Divergence Theorem (Gauss' Theorem)

Lesson 8: Measuring the Flow of a Vector Field ACROSS a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ACROSS*** the closed curve is measured by:

$$\oint_C \mathbf{Field(x,y) \bullet outerunitnormal\ ds}$$

$$= \int_a^b \mathbf{Field(x(t),y(t)) \bullet (y'(t),-x'(t))dt}$$

$$= \oint_C -\mathbf{n(x,y)dx + m(x,y)dy}$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left(\frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} \right) dx\ dy$$

Let $\mathbf{divField(x,y)} = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}.$

$$= \iint_R \mathbf{divField(x,y)\ dx\ dy}$$

Lesson 8 : The Flow of A Vector Field ACROSS a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_C \mathbf{Field(x,y)} \bullet \mathbf{outerunitnormal\ ds} = \iint_R \mathbf{divField(x,y)\ dx\ dy}$$

This measures the net flow of the vector field ***ACROSS*** the closed curve.

We define the divergence of the vector field as:

$$\mathbf{divField(x,y)} = \frac{\partial \mathbf{m}}{\partial \mathbf{x}} + \frac{\partial \mathbf{n}}{\partial \mathbf{y}} = \mathbf{D[m[x,y],x] + D[n[x,y],y]}$$

Lesson 8 : Measuring the Flow of a Vector Field ALONG a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ALONG*** the closed curve is measured by:

$$\oint_C \mathbf{Field(x,y) \bullet unittan\ ds}$$

$$= \int_a^b \mathbf{Field(x(t),y(t)) \bullet (x'(t),y'(t))dt}$$

$$= \oint_C \mathbf{m(x,y)dx + n(x,y)dy}$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dx\ dy$$

$$\text{Let } rotField(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}.$$

$$= \iint_R rotField(x,y)\ dx\ dy$$

Lesson 8 : The Flow of A Vector Field ALONG a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_{C} \mathbf{Field}(x,y) \bullet \mathbf{unittan}\, ds = \iint_{R} \mathbf{rotField}(x,y)\, dx\, dy$$

This measures the net flow of the vector field ***ALONG*** the closed curve.

We define the rotation of the vector field as:

$$\mathbf{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = D[n[x,y],x] - D[m[x,y],y]$$

Lesson 12: The Net Flow of A Vector Field ACROSS a Closed Surface:

Constructing a three-dimensional analog of using the Gauss-Green Theorem to compute the net flow of a vector field ***across*** a closed curve is not difficult. This is because the notion of divergence extends to three dimensions pretty naturally.

We will save the three-dimensional analog of flow ***ALONG*** for next chapter…

Lesson 12: The Net Flow of A Vector Field ACROSS a Closed Surface:

Let R be a solid in three dimensions with boundary surface (skin) C with no singularities on the interior region R of C. Then the net flow of the vector field Field(x,y,z) ***ACROSS*** the closed surface is measured by:

$$\oiint_C \mathbf{Field(x,y,z) \bullet outerunitnormal\ dA}$$

$$= \iiint_R \mathbf{divField(x,y,z)\ dx\ dy\ dz}$$

Let $\mathbf{Field(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big).}$

We define the <u>divergence</u> of the vector field as:

$$\mathbf{divField(x,y,z)} = \frac{\partial \mathbf{m}}{\partial \mathbf{x}} + \frac{\partial \mathbf{n}}{\partial \mathbf{y}} + \frac{\partial \mathbf{p}}{\partial \mathbf{z}}$$

$$= \mathbf{D[m[x,y,z],x] + D[n[x,y,z],y] + D[p[x,y,z],z]}$$

More Traditional Notation: The Divergence Theorem (Gauss' Theorem)

Let V be a solid in three dimensions with boundary surface (skin) S with no singularities on the interior region V of S. Then the **FLUX** of the vector field F(x,y,z) across the closed surface is measured by:

$$\oiint_S (\mathbf{F} \bullet \mathbf{n})\, dS = \iiint_V (\nabla \bullet \mathbf{F})\, dV$$

Let $F(x,y,z) = \left(m(x,y,z), n(x,y,z), p(x,y,z)\right)$.

Let $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ be known as "del", or the differential operator.

Note $\text{divField}(x,y,z) = \nabla \bullet F = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}$.

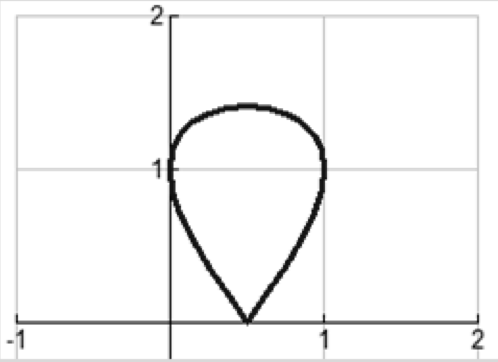
Finally, let n = outerunitnormal.

Example 1: Avoiding Computation Altogether

Let $\text{Field}(x,y) = (7x+2, y-6)$ **and let C be a closed curve given by**

$$C(t) = (x(t), y(t)) = \left(\sin^2(t), \cos(t)+\sin(t)\right) \text{ for } -\frac{\pi}{4} \le t \le \frac{3\pi}{4}.$$

Is the net flow of the vector field across the curve from inside to outside or outside to inside?



$$\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 7 + 1 = 8$$

$$\oint_C -n(x,y)dx + m(x,y)dy = \iint_R \text{divField}(x,y)\, dx\, dy = \iint_R 8\, dx\, dy$$

Since divField(x,y) is ALWAYS positive for all (x,y) and there are no singularities for any (x,y), this integral is positive for any closed curve.

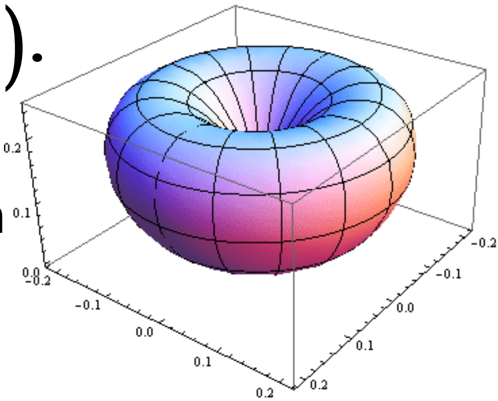
That is, for ANY closed curve, the net flow of the vector field across the curve is from inside to outside.

Example 2: Avoiding Computation Altogether

Let $\text{Field}(x,y,z) = \left(-x + y^2 - \cos(z), -y^3 + xz, -3z + 8x - 3e^y\right)$.

Let C be the a bounding surface of a solid region.

Is the net flow of the vector field across the surface from inside to outside or outside to inside?



$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = -1 - 3y^2 - 3 = -3y^2 - 4$$

Since Field(x, y, z) has no singularities inside R:

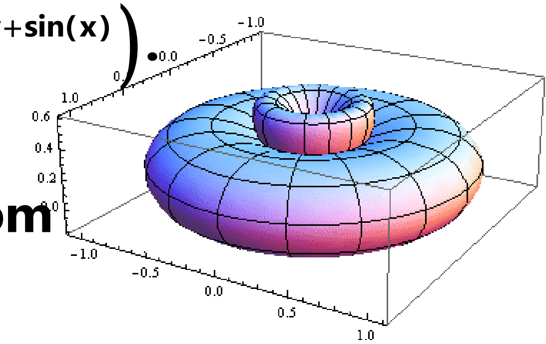
$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA = \iiint_R \text{divField}(x,y,z)\ dx\ dy\ dz$$

$$= \iiint_R -3y^2 - 4\ dx\ dy\ dz < 0$$

So for ANY closed surface, the net flow of the vector field across the surface is from outside to inside.

Example 3: Avoiding Computation Altogether

Let $\text{Field}(x,y,z) = \left(\sin(y) + z^5, zx^3\cos(x), \sin(5xy) - 3xe^{xy+\sin(x)}\right)$.



Let C be the bounding surface of a solid region.

Is the net flow of the vector field across the surface from inside to outside or outside to inside?

$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 0$$

Since Field(x, y, z) has no singularities inside R:

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\, dA = \iiint_R \text{divField}(x,y,z)\, dx\, dy\, dz$$

$$= \iiint_R 0\, dx\, dy\, dz = 0$$

That is, for ANY closed surface, the net flow of the vector field across the surface is 0.

Summary: The Divergence Locates Sources and Sinks

Let R be a solid in three dimensions with boundary surface (skin) C with no singularities on the interior region R of C. Then:

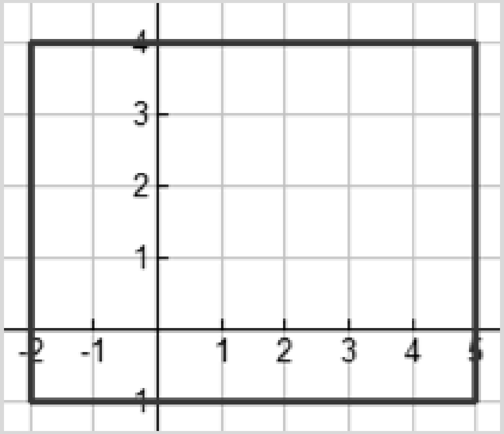
If divField(x,y,z)>0 for all points in C, then all these points are sources and the net flow of the vector field across C is from inside to outside.

If divField(x,y,z)<0 for all points in C, then all of these points are sinks and the net flow of the vector field across C is from outside to inside.

If divField(x, y, z) = 0 for all points in C, then the net flow of the vector field across C is 0.

Example 4: Find the Net Flow of a Vector Field ***ACROSS*** a Closed Curve

Let $\text{Field}(x, y) = \left(x^2 - 2xy, -y^2 + x\right)$ and let C be the rectangle bounded by $x = -2$, $x = 5$, $y = -1$, and $y = 4$. Measure the net flow of the vector field across the curve.



$$\text{divField}(x, y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 2x - 4y$$

$$\oint_C -n(x, y)dx + m(x, y)dy = \iint_R \text{divField}(x, y)\, dx\, dy$$

$$= \int_{-1}^{4}\int_{-2}^{5}\left(2x - 4y\right) dx\, dy$$

$$= -105$$

Negative. The net flow of the vector field across our closed curve is from outside to inside.

Example 5: Find the Net Flow of a Vector Field *ACROSS* a Closed Surface

Let $\text{Field}(x,y,z) = \left(2xy, -y^2, 5z + 4xz\right)$ and let C be the rectangular prism bounded by $-1 \le x \le 4, -2 \le y \le 3$, and $0 \le z \le 5$. Measure the net flow of the vector field across the closed surface.

$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 5 + 4x$$

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA = \iiint_R \text{divField}(x,y,z)\ dx\ dy\ dz$$

$$= \int_0^5 \int_{-2}^3 \int_{-1}^4 \left(5 + 4x\right) dx\ dy\ dz$$

$$= 1375$$

Positive. The net flow of the vector field across our closed surface is from inside to outside.

The Net Flow of a Vector Field Across an Open Surface

The Divergence Theorem is great for a closed surface, but it is not useful at all when your surface does not fully enclose a solid region. In this situation, we will need to compute a surface integral. For a parameterized surface, this is pretty straightforward:

$$\iint_C \mathbf{Field(x, y, z) \bullet outerunitnormal\ dA}$$

$$= \int_{t_1}^{t_2}\int_{s_1}^{s_2} \mathbf{Field(x(s,t), y(s,t), z(s,t)) \bullet normal(s,t)\ ds\ dt}$$

What is the normal vector??

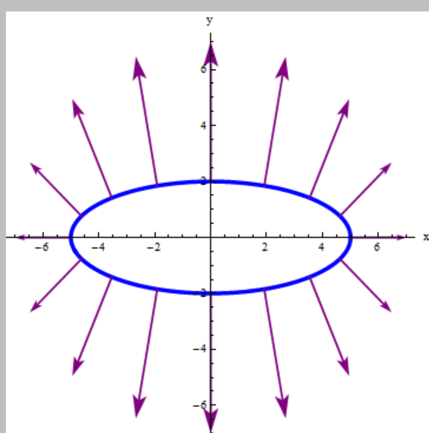
Normal Vectors: Curves Versus Surfaces

2-Dimensions:

$$\int_C \text{Field}(x,y) \bullet \text{outerunitnormal } ds$$

$$= \int_a^b \text{Field}(x(t),y(t)) \bullet \text{outernormal } dt$$

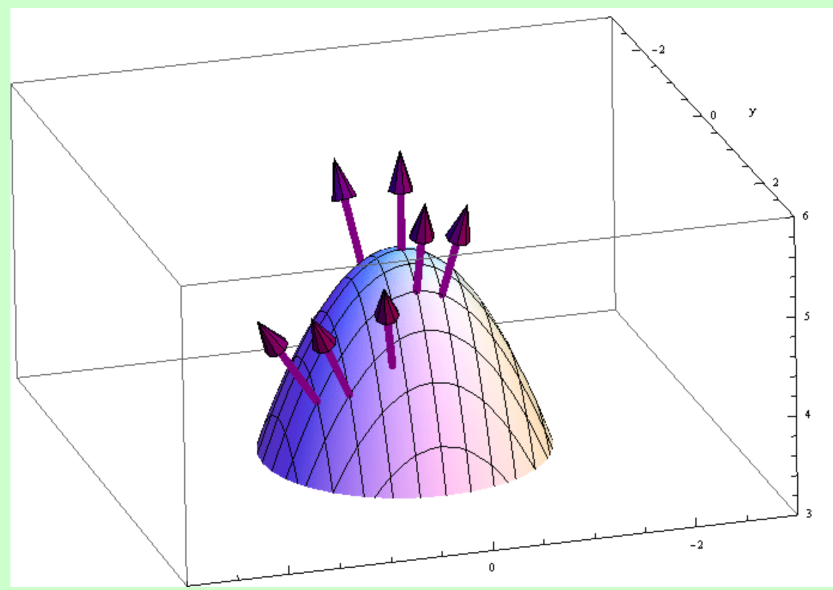
$$= \int_a^b \text{Field}(x(t),y(t)) \bullet (y'(t),-x'(t))dt$$



3-Dimensions:

$$\iint_C \text{Field}(x,y,z) \bullet \text{outernormal } dA$$

$$= \int_{t_1}^{t_2}\int_{s_1}^{s_2} \text{Field}(x(s,t),y(s,t),z(s,t)) \bullet \text{normal}(s,t)\, ds\, dt$$



In 2 dimensions, outernormal $= (y'(t),-x'(t))$.

This is more subtle in 3 dimensions...

Normal Vectors: Curves Versus Surfaces

$$\iint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\, dA = \int_{t_1}^{t_2}\int_{s_1}^{s_2} \text{Field}(x(s,t),y(s,t),z(s,t)) \bullet \text{normal}(s,t)\, ds\, dt$$

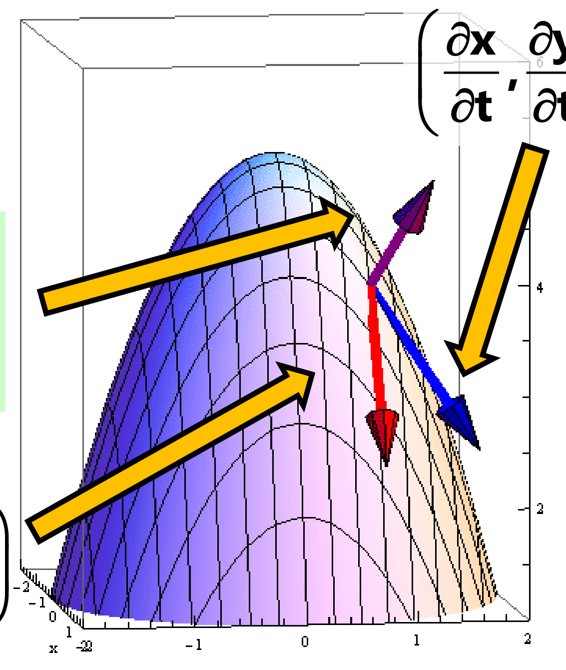
For a surface C parameterized by $(x(s,t),y(s,t),z(s,t))$, you can find two linearly-independent tangent vectors to the surface using partial derivatives:

$$\left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right) \text{ and } \left(\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right)$$

Use these two vectors tangent to the curve to generate your normal vector:

$$\text{normal}(s,t) = \left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right) \times \left(\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right)$$

$$\left(\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right)$$



$$\left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right)$$

Example 6: Using a Substitute Surface When the Divergence is 0

Let $\text{Field}(x,y,z) = \left(z+y, z-x, x^2\right)$.

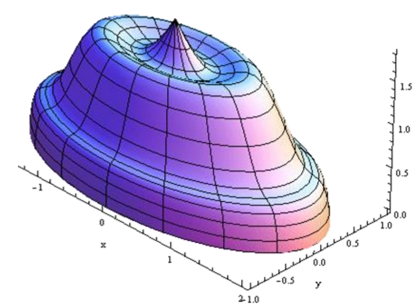
Let C be the bounding surface of the solid region pictured below, where C is the union of the pointy cap, C_1, and the elliptical base C_2. Find the net flow of the vector field across C_1.

$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 0$$

For $0 \le t \le 2\pi$ and $0 \le s \le \frac{\pi}{2}$:

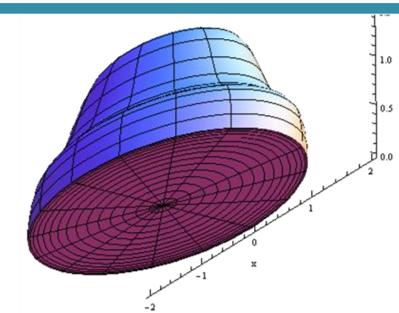
$$C_1: \left(x_1(s,t), y_1(s,t), z_1(s,t)\right)$$
$$= \left(2\sin(s)\cos(t), \sin(s)\sin(t), \frac{\cos(s)(1-\sin(8s))}{4} - s + \frac{\pi}{2}\right)$$

C_1



$$C_2: \left(x_2(s,t), y_2(s,t), z_2(s,t)\right)$$
$$= \left(2\sin(s)\cos(t), \sin(s)\sin(t), 0\right)$$

C_2

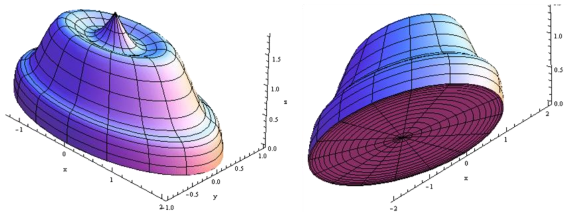


Example 6: Using a Substitute Surface When the Divergence is 0

Let $\text{Field}(x,y,z) = \left(z+y, z-x, x^2\right)$.

Let C be the bounding surface of the solid region, the union of the cap, C_1, and the elliptical base C_2.

Find the net flow of the vector field across C_1.



$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 0$$

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA = \iiint_R \text{divField}\ dx\ dy\ dz = 0$$

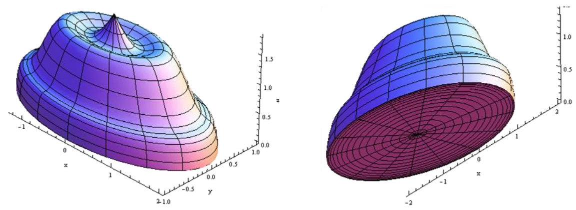
So the net flow of the vector field across the closed surface C is 0. However, this calculation does NOT imply that the net flow of the vector field across C_1 or C_2 is 0.

Example 6: Using a Substitute Surface When the Divergence is 0

Let $\text{Field}(x, y, z) = \left(z + y, z - x, x^2\right)$.

Let C be the bounding surface of the solid region, the union of the cap, C_1, and the elliptical base C_2.

Find the net flow of the vector field across C_1.



$$0 = \iiint_R \text{divField}(x, y, z)\ dx\ dy\ dz$$

$$= \oiint_C \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA$$

$$= \iint_{C_1} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA - \iint_{C_2} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA$$

The next slide has an (optional) explanation of why this is not addition!

So $\iint_{C_1} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA = \iint_{C_2} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA$.

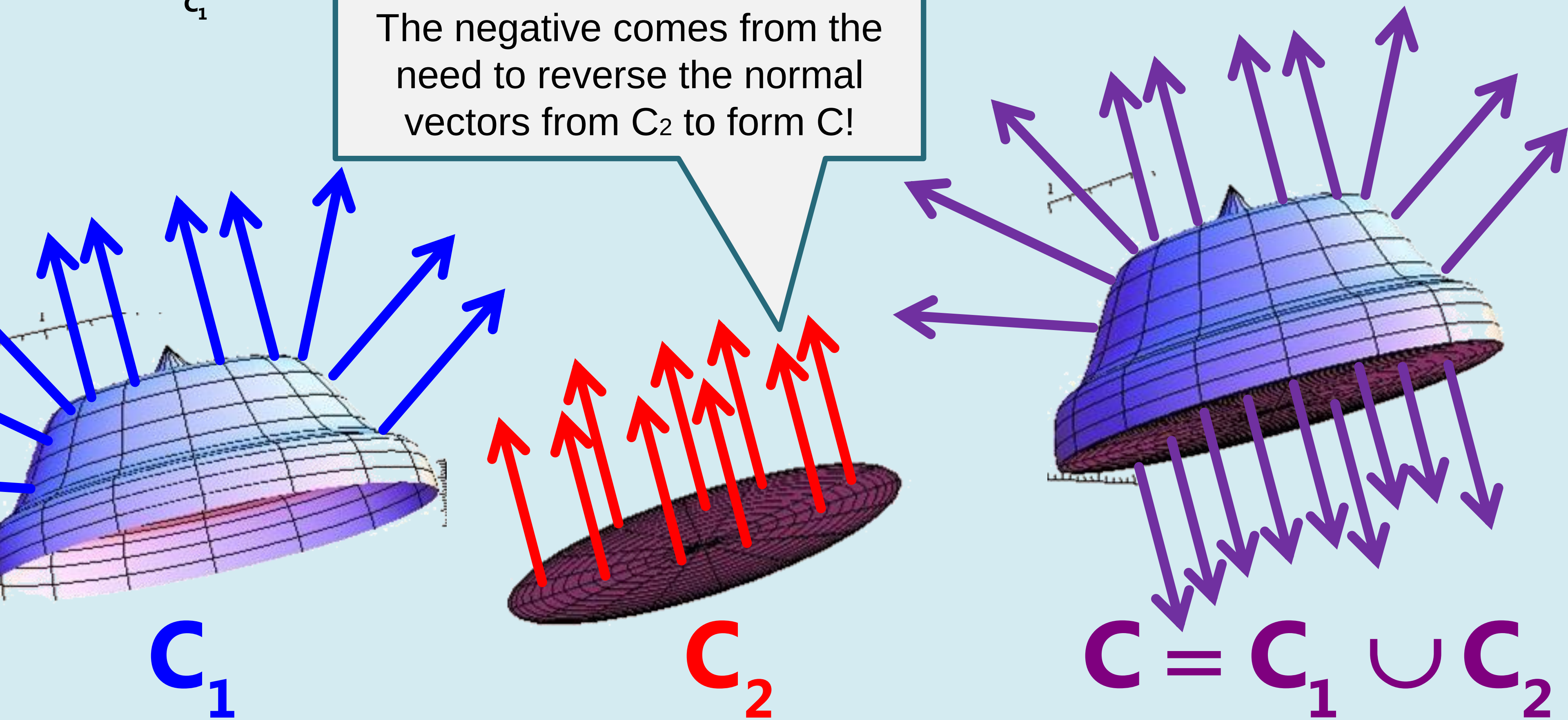
Since C_2 is easier to work with, we'll use this substitute surface instead!

Optional Slide: Why the "Negative" in Example 6?

$$\oiint_{C} \mathbf{Field}(x,y,z) \bullet \mathbf{outerunitnormal}\ dA$$

$$= \iint_{C_1} \mathbf{Field}(x,y,z) \bullet \mathbf{outerunitnormal}\ dA - \iint \mathbf{Field}(x,y,z) \bullet \mathbf{outerunitnormal}\ dA$$

The negative comes from the need to reverse the normal vectors from C_2 to form C!



$$\iint_{C_2} \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \text{Field}(x(s,t), y(s,t), z(s,t)) \bullet \text{normal}(s,t)\ ds\ dt$$

$C_2: \left(x_2(s,t), y_2(s,t), z_2(s,t)\right) = \left(2\sin(s)\cos(t), \sin(s)\sin(t), 0\right)$ for $0 \le s \le \frac{\pi}{2}$ and $0 \le t < 2\pi$

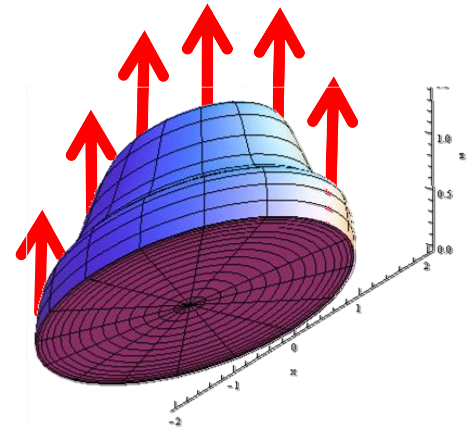
$$\text{normal}(s,t) = \left(\frac{\partial x_2}{\partial s}, \frac{\partial y_2}{\partial s}, \frac{\partial z_2}{\partial s}\right) \times \left(\frac{\partial x_2}{\partial t}, \frac{\partial y_2}{\partial t}, \frac{\partial z_2}{\partial t}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial x_2}{\partial s} & \frac{\partial y_2}{\partial s} & \frac{\partial z_2}{\partial s} \\ \frac{\partial x_2}{\partial t} & \frac{\partial y_2}{\partial t} & \frac{\partial z_2}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ 2\cos(s)\cos(t) & \cos(s)\sin(t) & 0 \\ -2\sin(s)\sin(t) & \sin(s)\cos(t) & 0 \end{vmatrix}$$

$$= 0i - 0j + (2\sin(s)\cos(s)\cos^2(t) + 2\sin(s)\cos(s)\sin^2(t))k$$

$$= (0, 0, 2\sin(s)\cos(s))$$

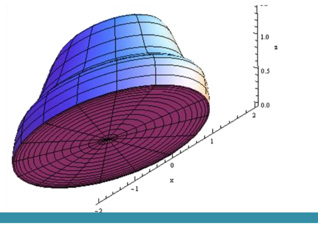


These normals point in the correct direction because from $0 \le s \le \frac{\pi}{2}$, $(0, 0, 2\sin(s)\cos(s))$ points up out of the elliptical base.

Example 6: Using a Substitute Surface When the Divergence is 0

$$\text{Field}(x,y,z) = \left(z+y,\, z-x,\, x^2\right)$$

$$C_2: \left(x_2(s,t), y_2(s,t), z_2(s,t)\right) = \left(2\sin(s)\cos(t), \sin(s)\sin(t), 0\right)$$



$$\iint_{C_2} \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \text{Field}(x(s,t), y(s,t), z(s,t)) \bullet \text{normal}(s,t)\ ds\ dt$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\sin(s)\sin(t), -2\cos(t)\sin(s), 4\cos^2(t)\sin^2(s)\right) \bullet (0,0,2\sin(s)\cos(s))\ ds\ dt$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3(s)\cos^2(t)\cos(s)\ ds\ dt$$

$$= 2\pi$$

The net flow of the vector field across C_1 is with the direction of the normal vectors (down to up).

Summary: Using a Substitute Surface When the Divergence is 0

Let C be the bounding surface of a solid region such that $C = C_1 \cup C_2$ for two open surfaces C_1 and C_2. Let Field(x,y,z) be a vector field with no singularities contained within C such that $\text{divField}(x, y, z) = 0$ away from singularities. Then:

$$\iint_{C_1} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA = \iint_{C_2} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA$$

This allows us to substitute C_1 for C_2 or vice-versa when computing a surface integral. Trade a crazy surface for a simpler one!

Note : Just because $\iint_{C} \text{Field}(x, y, z) \bullet \text{outerunitnormal}\ dA = 0$, that says nothing about C_1 or C_2.

Lesson 8: Net Flow Across When divField(x,y)=0

Let $\text{divField}(x,y)=\frac{\partial m}{\partial x}+\frac{\partial n}{\partial y}=0$**. Here are some conclusions about the net flow of the vector field across various closed curves:**

If C doesn't contain any singularities, then $\oint_C -n(x,y)dx+m(x,y)dy=0$**.**

If C contains a singularity, then $\oint_C -n(x,y)dx+m(x,y)dy=\oint_{C_1} -n(x,y)dx+m(x,y)dy$ **for any substitute curve** C_1 **containing the same singularity (and no new extras).**

If C contains n singularities, then

$$\oint_C -n(x,y)dx+m(x,y)dy=\oint_{C_1} -n(x,y)dx+m(x,y)dy+\ldots+\oint_{C_n} -n(x,y)dx+m(x,y)dy$$

for little circles, $C_1,\ldots,C_n$**, encapsulating each of these singularities.**

Lesson 12: Net Flow Across When divField(x,y,z)=0

Let $\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 0$. Here are some conclusions about the net flow of the vector field across various closed surfaces:

If C doesn't contain any singularities, then $\oiint_C \text{Field}(x,y,z) \bullet \text{outernormal}\ dA = 0$.

If C contains a singularity, then

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA = \oiint_{C_1} \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA$$

for any substitute surface C_1 containing the same singularity (and no extras).

If C contains n singularities, then

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA$$

$$= \oiint_{C_1} \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA + \ldots + \oiint_{C_n} \text{Field}(x,y,z) \bullet \text{outerunitnormal}\ dA$$

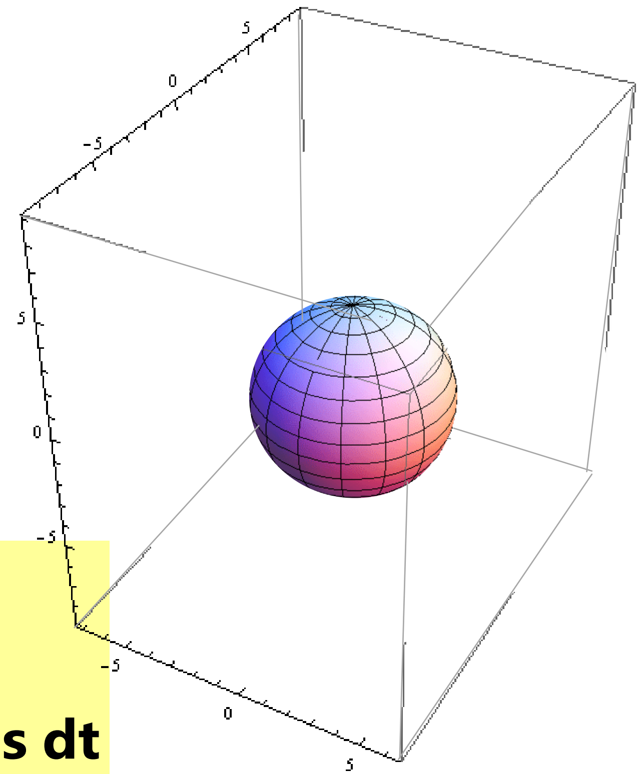
for little spheres, $C_1, \ldots, C_n$, encapsulating each of these singularities.

Example 7: Using a Substitute Surface With Singularities (Details in Mathematica Notebook)

Let $\text{Field}(x,y,z) = \left(\dfrac{x}{(x^4+y^4+z^4)^{3/4}}, \dfrac{y}{(x^4+y^4+z^4)^{3/4}}, \dfrac{z}{(x^4+y^4+z^4)^{3/4}} \right)$

and let C be the boundary to the region pictured at the right.

Find the net flow of the vector field across C.



$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z} = 0,$$

but we have a singularity at (0,0,0).

Replace the surface with a small sphere centered at (0,0,0):

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outerunitnormal}\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \text{Field}(x(s,t), y(s,t), z(s,t)) \bullet \text{normal}(s,t)\, ds\, dt$$

Example 7: Using a Substitute Surface With Singularities (Details in Mathematica Notebook)

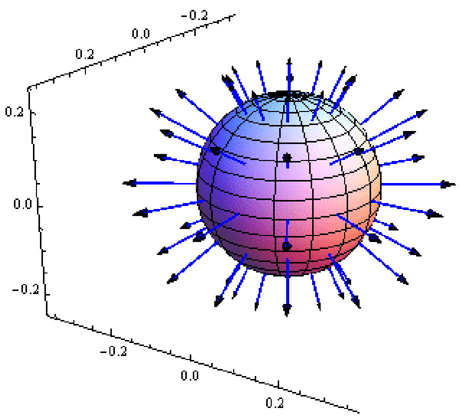
Find normal(s, t) :

```
Clear[x, y, z, s, t, sphere, normal];
singularity = {0, 0, 0};
radius = 0.2;
{x[s_, t_], y[s_, t_], z[s_, t_]} = singularity + {radius Sin[s] Cos[t], radius Sin[s] Sin[t], radius Cos[s]};
normal[s_, t_] = TrigExpand[D[{x[s, t], y[s, t], z[s, t]}, s]×D[{x[s, t], y[s, t], z[s, t]}, t]]
```

$\{0. + 0.02 \mathrm{Cos}[t] - 0.02 \mathrm{Cos}[s]^2 \mathrm{Cos}[t] + 0.02 \mathrm{Cos}[t] \mathrm{Sin}[s]^2,\ 0. + 0.02 \mathrm{Sin}[t] - 0.02 \mathrm{Cos}[s]^2 \mathrm{Sin}[t] + 0.02 \mathrm{Sin}[s]^2 \mathrm{Sin}[t],\ 0. + 0.04 \mathrm{Cos}[s] \mathrm{Sin}[s]\}$

Verify they are OUTERnormals:

```
Show[sphere, Table[Vector[normal[s, t], Tail → {x[s, t], y[s, t], z[s, t]}, ScaleFactor → 4], {s, 0, π, π/6}, {t, 0, 2 π, π/6}], Boxed → False,
  ViewPoint → CMView, PlotRange → All]
```



Example 7: Using a Substitute Surface With Singularities (Details in Mathematica Notebook)

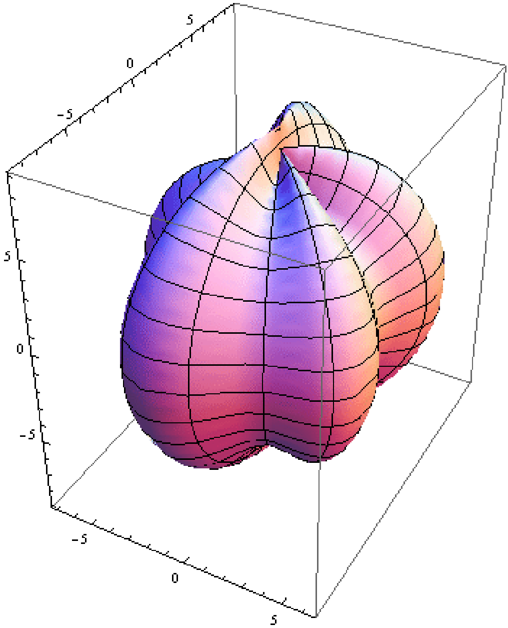
Let $\text{Field}(x,y,z) = \left(\dfrac{x}{(x^4+y^4+z^4)^{3/4}}, \dfrac{y}{(x^4+y^4+z^4)^{3/4}}, \dfrac{z}{(x^4+y^4+z^4)^{3/4}}\right)$

and let C be the boundary to the region pictured at the right.

Find the net flow of the vector field across C.

$$\int_0^{2\pi}\int_0^{\pi} \text{Field}(x(s,t), y(s,t), z(s,t)) \bullet \text{normal}(s,t)\, ds\, dt = 19.446$$

So the net flow of the vector field across the wavy surface (and the sphere) is inside to outside.



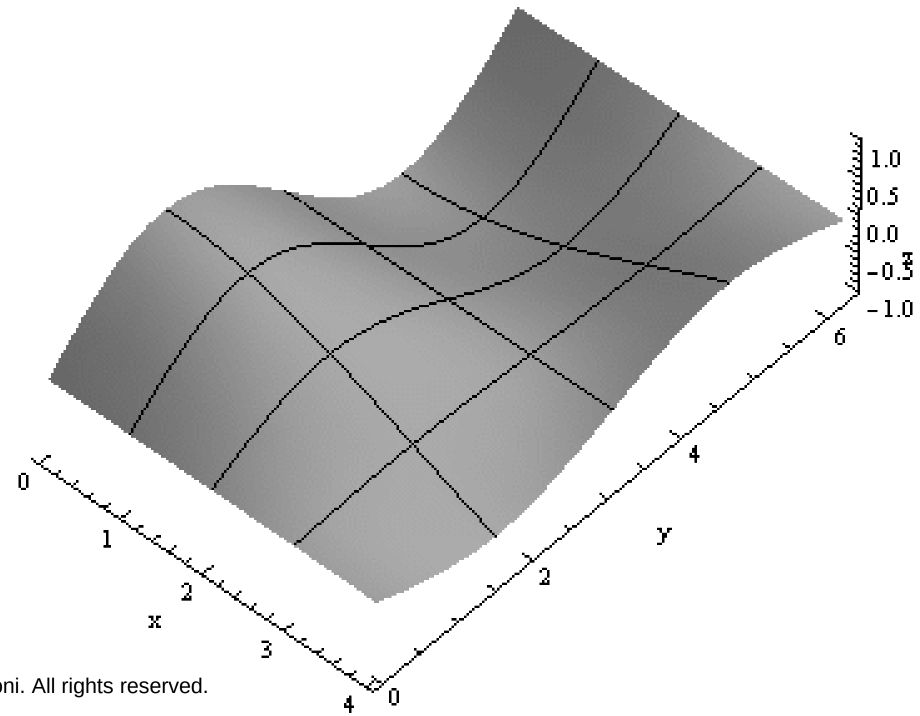
Example 8: Surface Area

Consider the two-dimensional surface in xyz-space described by the equation $f(x,y)=\sin(y)\cos\left(\frac{x}{2}\right)$. Find the surface area of the surface given the bounds $0\le x\le 4$ and $0\le y\le 2\pi$:

First, we come up with a parameterization of the surface:

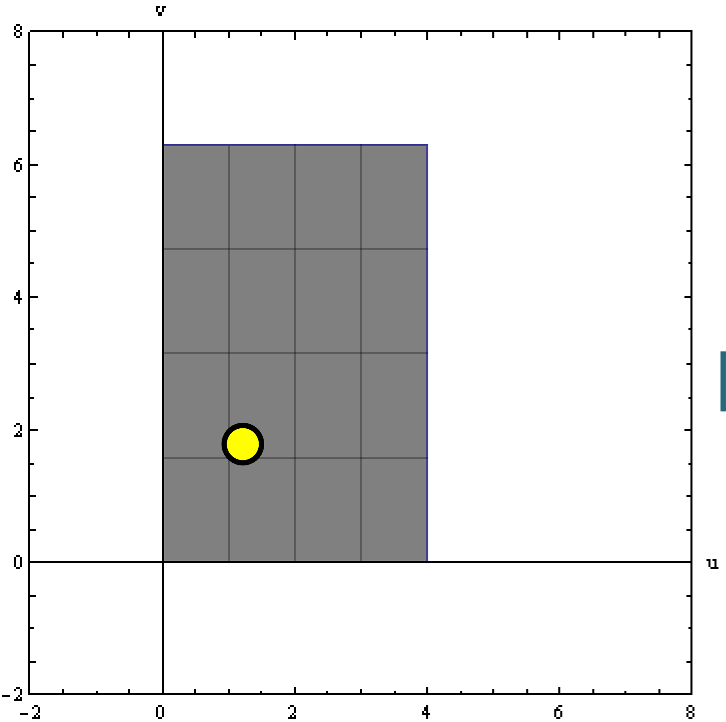
$$x(u,v)=u$$
$$y(u,v)=v$$
$$z(u,v)=\sin(v)\cos\left(\frac{u}{2}\right)$$

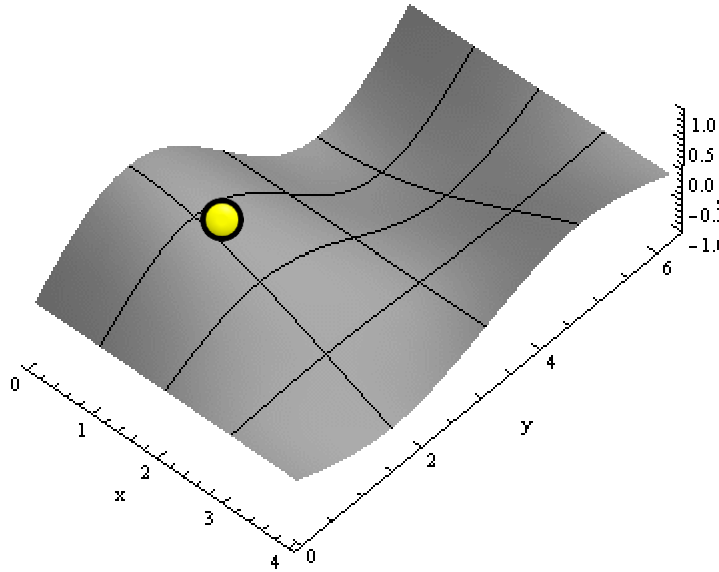
$$0\le u\le 4$$
$$0\le v\le 2\pi$$



The Surface Area Conversion Factor

Now we can think of this as a map from a uv-rectangle to a xyz-surface.

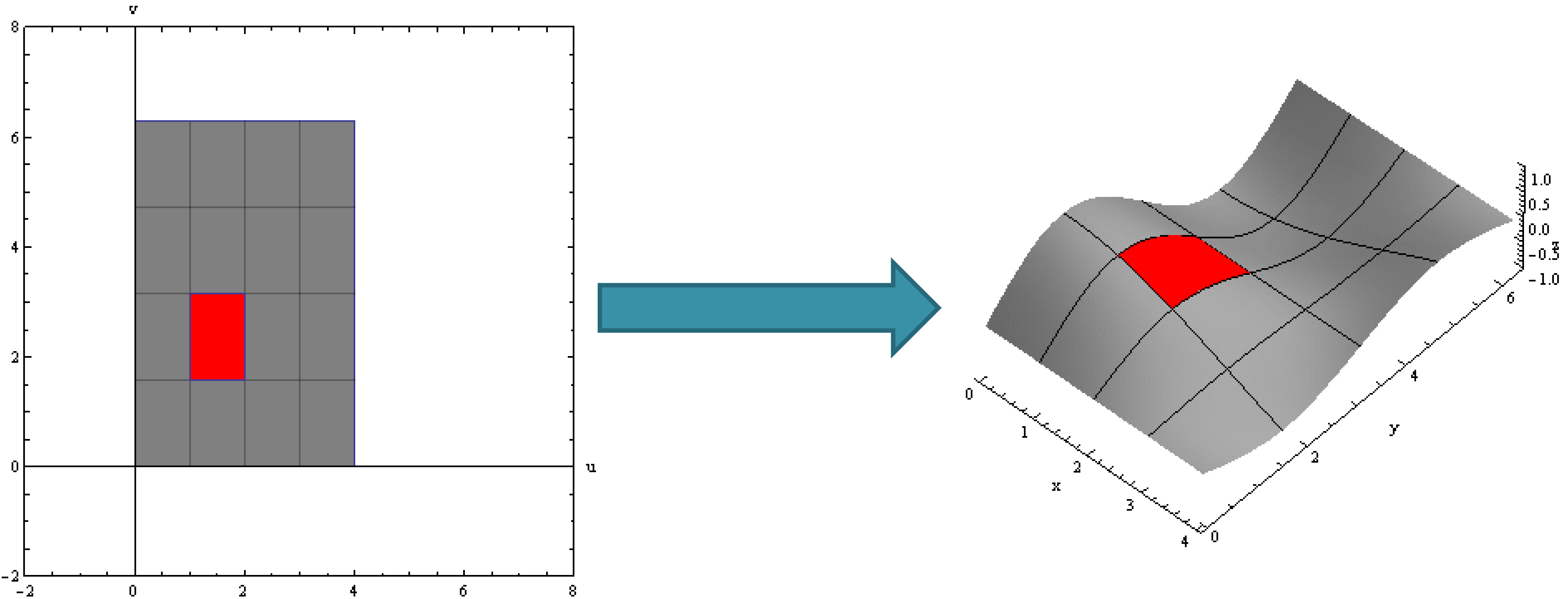




$$(u, v) \longrightarrow \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

The Surface Area Conversion Factor

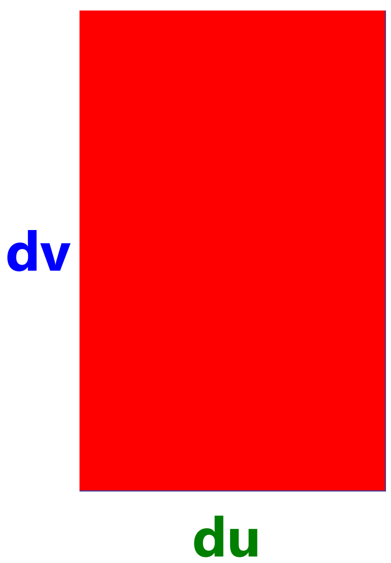
We can also consider how a small uv-rectangle maps into xyz-space:



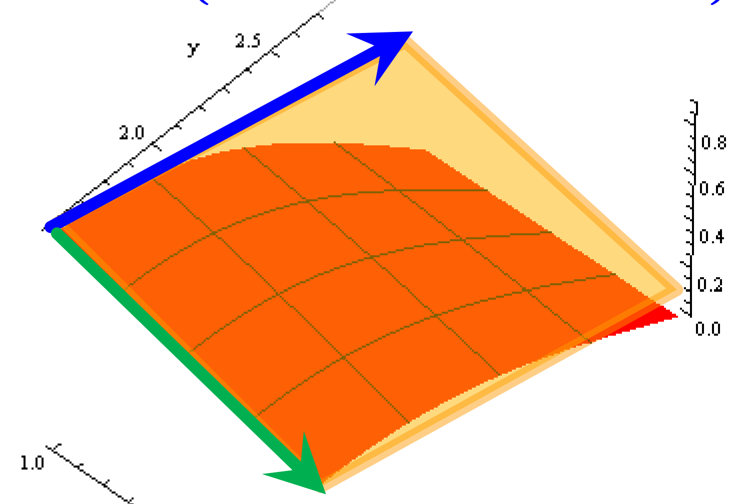
As in previous chapters, we'll relate a small change in area on the uv-rectangular region relates to a change in surface area on the xyz-surface . Notice that uv-rectangles of fixed area map into little xyz-surfaces of varying surface area.

The Surface Area Conversion Factor

Imagine a tiny rectangle in the uv-plane:



$$\left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v}\right)$$

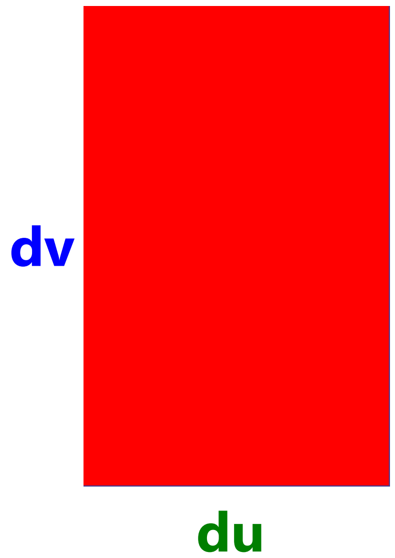


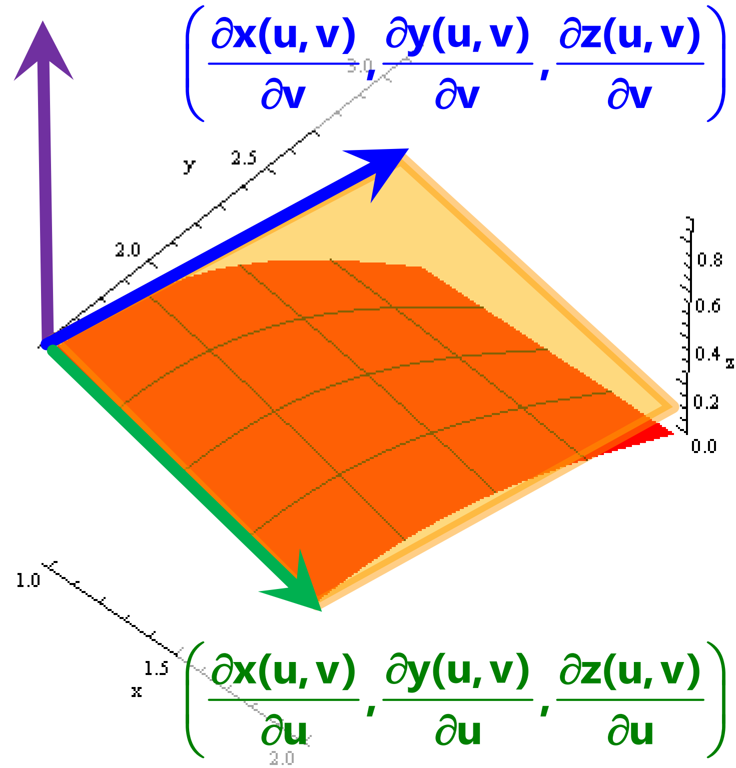
$$\left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u}\right)$$

The change in surface area that results with the xyz-surface can be approximated by the area of the parallelogram generated by the tangent vectors given by taking the partial derivative of the map (x(u,v),y(u,v),z(u,v)) with respect to u and v.

The Surface Area Conversion Factor

$$\left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u}\right) \times \left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v}\right)$$



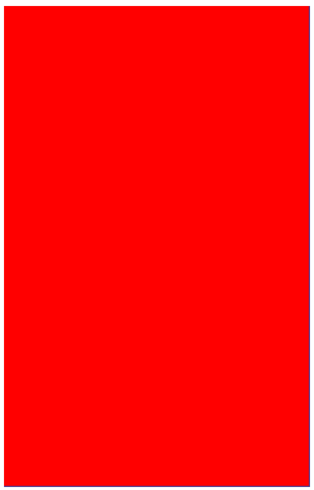


Now recall that the area of a parallelogram in 3D-space can be quickly computed by finding the magnitude of the cross product of its generating vectors. This is equivalent to the length of the normal vector shown above.

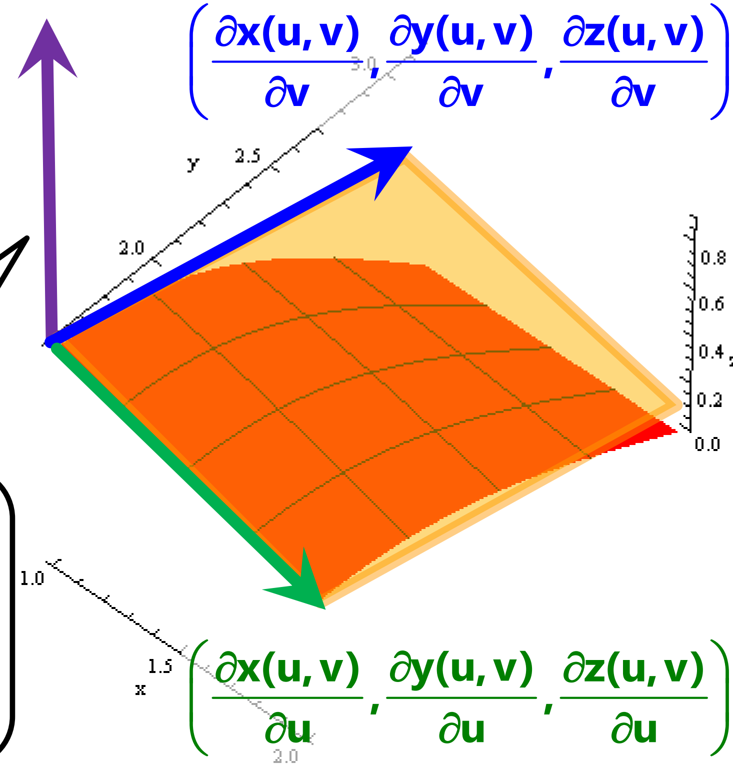
The Surface Area Conversion Factor

$$\left(\frac{\partial x(u,v)}{\partial u},\frac{\partial y(u,v)}{\partial u},\frac{\partial z(u,v)}{\partial u}\right)\times\left(\frac{\partial x(u,v)}{\partial v},\frac{\partial y(u,v)}{\partial v},\frac{\partial z(u,v)}{\partial v}\right)$$

dv



du



$$\left(\frac{\partial x(u,v)}{\partial v},\frac{\partial y(u,v)}{\partial v},\frac{\partial z(u,v)}{\partial v}\right)$$

$$\left(\frac{\partial x(u,v)}{\partial u},\frac{\partial y(u,v)}{\partial u},\frac{\partial z(u,v)}{\partial u}\right)$$

A longer normal vector means that the uv-rectangle has much more surface area in xyz-space (more curvature, really)

$$SA_{xyz}(u,v)=\left|\left(\frac{\partial x(u,v)}{\partial u},\frac{\partial y(u,v)}{\partial u},\frac{\partial z(u,v)}{\partial u}\right)\times\left(\frac{\partial x(u,v)}{\partial v},\frac{\partial y(u,v)}{\partial v},\frac{\partial z(u,v)}{\partial v}\right)\right|$$

The Surface Area Conversion Factor

Given a parameterization of a surface (x(u,v),y(u,v),z(u,v)), you can find the surface area of this surface for $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$ by integrating the following:

$$SA_{xyz}(u,v) = \left| \left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u} \right) \times \left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v} \right) \right|$$

If you write $T(u,v) = (x(u,v), y(u,v), z(u,v))$, you can write this a bit more concisely as the following:

$$\begin{aligned} SA_{xyz}(u,v) &= \left| \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right| \\ &= \sqrt{\left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right) \bullet \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right)} \end{aligned}$$

The Surface Area Conversion Factor

Given a parameterization of a surface (x(u,v),y(u,v),z(u,v)), you can find the surface area of this surface for $u_1 \le u \le u_2$ and $v_1 \le v \le v_2$ by integrating the following:

$$\iint_R dA = \int_{v_1}^{v_2}\int_{u_1}^{u_2} SA_{xyz}(u,v)\,du\,dv$$

Let $T(u,v) = (x(u,v), y(u,v), z(u,v))$ be a map from uv-space to xyz-space:

Then $SA_{xyz}(u,v) = \left|\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right|$

$$= \sqrt{\left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right) \bullet \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right)}$$

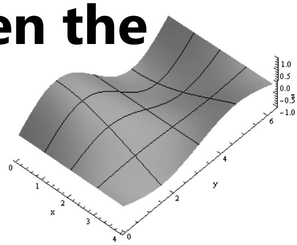
Is this reminiscent of the arc length formula from last year? It should be. Here it is:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Challenge: Find the connection!!

Example 8 Revisited: Surface Area

Find the surface area of the surface $f(x, y) = \sin(y)\cos\left(\frac{x}{2}\right)$ given the bounds $0 \le x \le 4$ and $0 \le y \le 2\pi$:



Let $T(u, v) = (x(u, v), y(u, v), z(u, v)) = \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$ for $0 \le u \le 4$ and $0 \le v \le 2\pi$.

Mathematica calculates

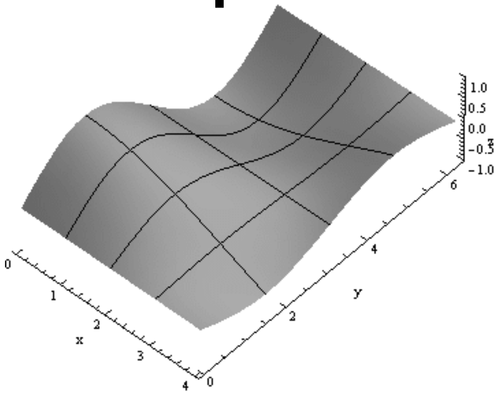
$$SA_{xyz}(u, v) = \sqrt{\left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right) \bullet \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right)}$$

$$= \frac{1}{4}\sqrt{3(7 + \cos(u)) + (3 + 5\cos(u))\cos(2v)}$$

Use Mathematica again: $\int_0^{2\pi}\int_0^4 SA_{xyz}(u, v)\, du\, dv \approx 28.298$

Example 9: Surface Integrals

Consider the parameterized surface below for $0 \le u \le 4$ and $0 \le v \le 2\pi$:



$$T(u,v) = \left(x(u,v), y(u,v), z(u,v)\right) = \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

The surface is made of a mixture of various metals of varying density described by $g(x,y,z) = \left|xy - z\right|$ g / cm^2. Find the mass of the surface.

Use Mathematica and the previously computed $SA_{xyz}(u,v)$:

$$\int_0^{2\pi}\int_0^4 g(x(u,v), y(u,v), z(u,v)) SA_{xyz}(u,v)\, du\, dv \approx 174.221 \text{ g}$$

Summary: Surface Integrals:

Given a parameterization of a surface (x(u,v),y(u,v),z(u,v)), you can find the surface integral of the function g(x,y,z) for $u_1 \le u \le u_2$ and $v_1 \le v \le v_2$ with respect to surface area by integrating the following:

$$\iint_R g(x,y,z)\,dA = \int_{v_1}^{v_2}\int_{u_1}^{u_2} g(x(u,v),y(u,v),z(u,v))\, SA_{xyz}(u,v)\, du\, dv$$

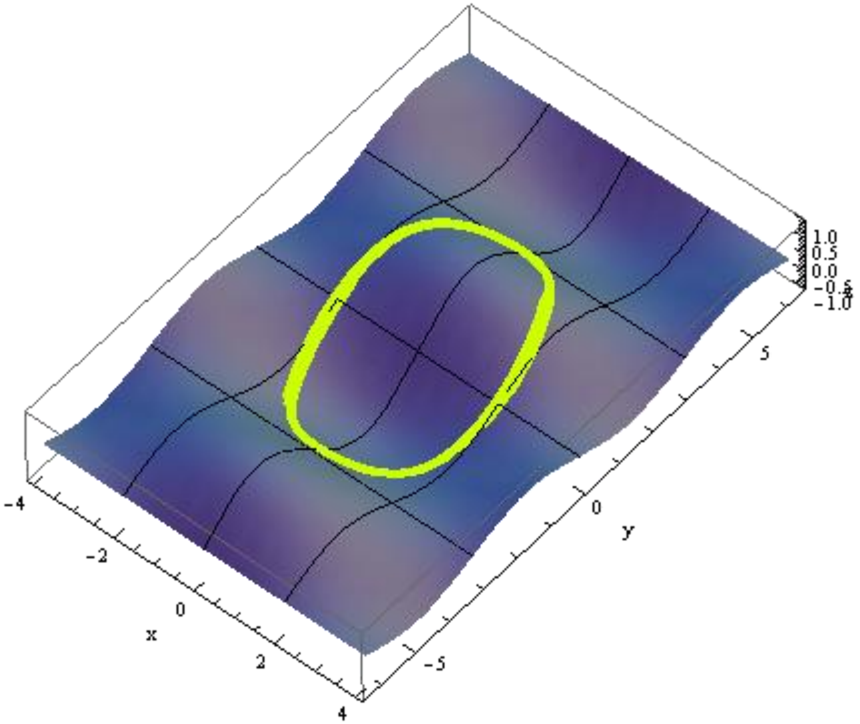
Let $T(u,v) = (x(u,v),y(u,v),z(u,v))$ be a map from uv-space to xyz-space:

$$\text{Then } SA_{xyz}(u,v) = \left|\frac{\partial T}{\partial u}\times\frac{\partial T}{\partial v}\right|$$

$$= \sqrt{\left(\frac{\partial T}{\partial u}\times\frac{\partial T}{\partial v}\right)\bullet\left(\frac{\partial T}{\partial u}\times\frac{\partial T}{\partial v}\right)}$$

Example 10: Surface Area Enclosed in a Curve Plotted on a Surface

Find the area of the elliptical region (2cos(t),3sin(t)) on this same surface:



$$T(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$$
$$= \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

The filled in ellipse is given by:

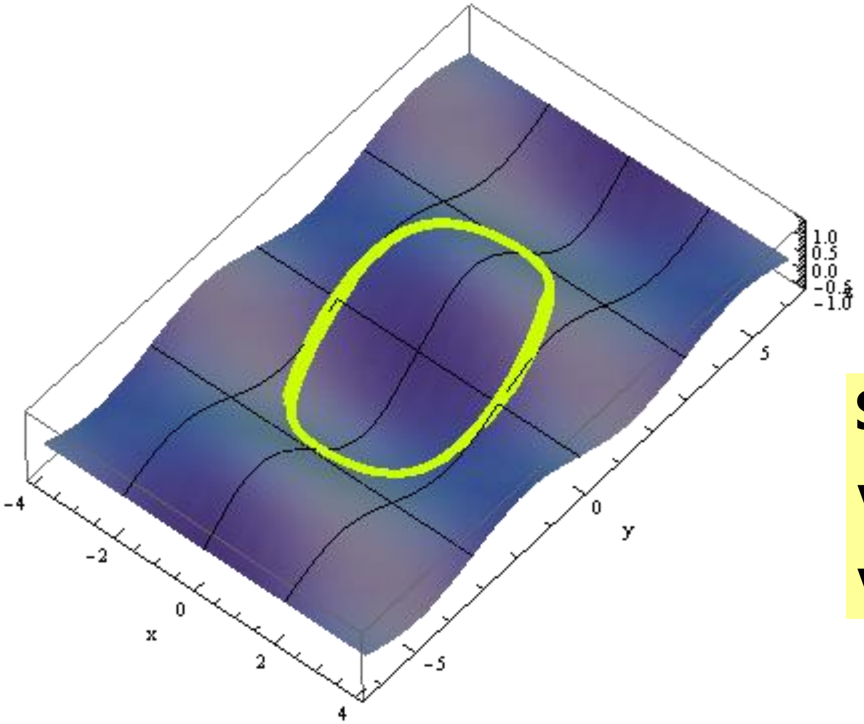
$$(2s\cos(t), 3s\sin(t))$$

for $0 \le t \le 2\pi$ **and** $0 \le s \le 1$.

To map it onto the surface, let $u(s,t) = 2s\cos(t)$ **and** $v(s,t) = 3s\sin(t)$, **and plot** $\left(x(u(s,t), v(s,t)), y(u(s,t), v(s,t)), z(u(s,t), v(s,t))\right)$.

Example 10: Surface Area Enclosed in a Curve Plotted on a Surface

Find the area of the elliptical region (2cos(t),3sin(t)) on this same surface:



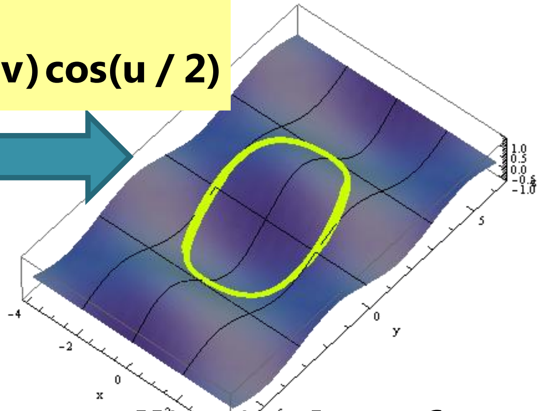
$$T(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$$
$$= \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

So we are have a second change of variables with $u(s,t) = 2s\cos(t)$ and $v(s,t) = 3s\sin(t)$!

$$\iint_{\text{ellipse}} SA_{xyz}(u,v)\,du\,dv = \int_0^{2\pi}\int_0^1 SA_{xyz}(u(s,t), v(s,t))\left|A_{uv}(s,t)\right|\,ds\,dt$$

Example 10: Surface Area Enclosed in a Curve Plotted on a Surface

st-rectangle → $\begin{cases} u(s,t) = 2s\cos(t) \\ v(s,t) = 3s\sin(t) \end{cases}$ → **uv-ellipse** → $\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = \sin(v)\cos(u/2) \end{cases}$ →



xyz-elliptical surface

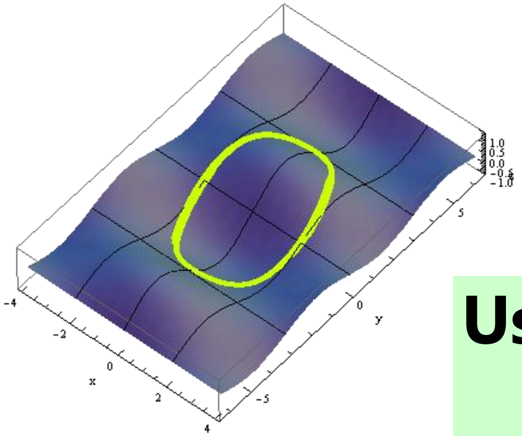
So we need $SA_{xyz}(u,v)$ to do our xyz-integral in uv-space,

and we need $\left|A_{uv}(s,t)\right|$ to do our uv-integral in st-space.

$$\iint_{\text{ellipse}} SA_{xyz}(u,v)\,du\,dv = \int_0^{2\pi}\int_0^1 SA_{xyz}(u(s,t),v(s,t))\left|A_{uv}(s,t)\right|\,ds\,dt$$

Example 10: Surface Area Enclosed in a Curve Plotted on a Surface

Find the area of the elliptical region (2cos(t),3sin(t)) on this same surface:



$$T(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$$

$$= \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

Using $u(s,t) = 2s\cos(t)$ and $v(s,t) = 3s\sin(t)$,

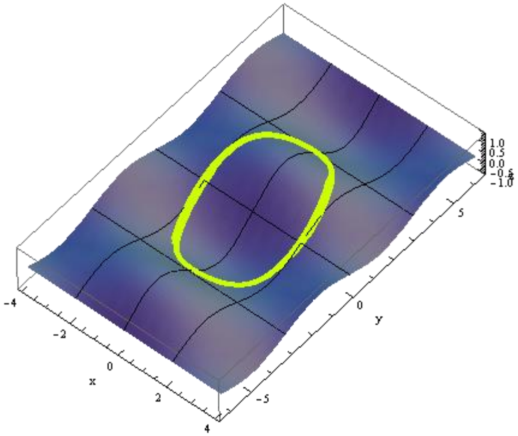
$$A_{uv}(s,t) = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} 2\cos(t) & 3\sin(t) \\ -2s\sin(t) & 3s\cos(t) \end{vmatrix}$$

$$= 6s$$

Example 10: Surface Area Enclosed in a Curve Plotted on a Surface (Bonus Material!)

Find the area of the elliptical region (2cos(t),3sin(t)) on this same surface:



$$T(u,v) = \left(x(u,v), y(u,v), z(u,v)\right) = \left(u, v, \sin(v)\cos\left(\frac{u}{2}\right)\right)$$

Letting Mathematica polish this one off, we get:

$$\int_0^{2\pi}\int_0^1 SA_{xyz}(u(s,t),v(s,t))\left|A_{uv}(s,t)\right| ds\, dt = \int_0^{2\pi}\int_0^1 6s\, SA_{xyz}(u(s,t),v(s,t))\, ds\, dt$$

$$\approx 22.0667$$

Scalar Surface Integrals versus Vector Surface Integrals

Vector Surface Integral: Let R be a solid in three dimensions with boundary surface (skin) C with no singularities on the interior region R of C. Then the net flow of the vector field Field(x,y,z) ***ACROSS*** the closed surface is measured by:

$$\oiint_C \text{Field}(x,y,z) \bullet \text{outernormal}\ dA = \iiint_R \text{divField}(x,y,z)\ dx\ dy\ dz$$

$$= \iiint_R \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}\ dx\ dy\ dz$$

Scalar Surface Integral: The Divergence Theorem is great for a closed surface, but it is not useful at all when your surface does not fully enclose a solid region. In this situation, we will need to compute a surface integral:

$$\iint_R g(x,y,z)dA$$

$$= \int_{v_1}^{v_2}\int_{u_1}^{u_2} g(x(u,v),y(u,v),z(u,v))\ SA_{xyz}(u,v)\ du\ dv$$