Lesson 13

# Stokes' Theorem, Curl, 3D Flow Along

# Lesson 8: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ACROSS*** the closed curve is measured by:

$$\oint_C \mathbf{Field(x,y) \bullet outerunitnormal\ ds}$$

$$= \int_a^b \mathbf{Field(x(t),y(t)) \bullet (y'(t),-x'(t))dt}$$

$$= \oint_C \mathbf{-n(x,y)dx + m(x,y)dy}$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left( \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} \right) dx\ dy \qquad \text{Let } divField(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}.$$

$$= \iint_R divField(x,y)\ dx\ dy$$

# Lesson 8 : The Net Flow of A Vector Field ACROSS a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_C \mathbf{Field(x,y)} \bullet \mathbf{outerunitnormal\ ds} = \iint_R \mathbf{divField(x,y)\ dx\ dy}$$

This measures the net flow of the vector field ***ACROSS*** the closed curve.

**We define the divergence of the vector field as:**

$$\mathbf{divField(x,y)} = \frac{\partial \mathbf{m}}{\partial \mathbf{x}} + \frac{\partial \mathbf{n}}{\partial \mathbf{y}} = \mathbf{D[m[x,y],x] + D[n[x,y],y]}$$

# Lesson 12: The Net Flow of A Vector Field ACROSS a Closed Surface:

Let R be a solid in three dimensions with boundary surface (skin) C with no singularities on the interior region R of C. Then the net flow of the vector field Field(x,y,z) ***ACROSS*** the closed surface is measured by:

$$\oiint_{C} \text{Field}(x,y,z) \bullet \text{outernormal}\, dA$$

$$= \iiint_{R} \text{divField}(x,y,z)\, dx\, dy\, dz$$

Let $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$.

We define the divergence of the vector field as:

$$\text{divField}(x,y,z) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}$$

$$= D[m[x,y,z],x] + D[n[x,y,z],y] + D[p[x,y,z],z]$$

# Lesson 12: The Divergence Theorem (Using Traditional Notation)

Let V be a solid in three dimensions with boundary surface (skin) S with no singularities on the interior region V of S. Then the net flow of the vector field F(x,y,z) ***ACROSS*** the closed surface is measured by:

$$\oiint_S (\mathbf{F} \bullet \mathbf{n})\, dS = \iiint_V (\nabla \bullet \mathbf{F})\, dV$$

**Let** $\mathbf{F(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big).}$

**Let** $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ **be known as "del", or the differential operator.**

**Note** $\mathbf{divField(x,y,z)} = \nabla \bullet \mathbf{F} = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}.$

**Finally, let n = outerunitnormal.**

# Lesson 13: The Net Flow of A Vector Field ALONG a Curve

Lesson 12 was about constructing a three-dimensional analog of using the Gauss-Green Theorem to compute the net flow of a vector field **ACROSS** a **SURFACE**. All we did was upgrade to a surface, and extend the definition of divergence to three dimensions.

Lesson 13 is all about constructing a three-dimensional analog of the net flow of a vector field **ALONG** a **CURVE**.

Let's begin by reviewing how we did this in 2-dimensions for an **OPEN** curve:

# Lesson 8: Measuring the Net Flow of a Vector Field ALONG an OPEN Curve in 2-Dimensions

Recall that if C is an open curve, then we can't use the Gauss-Green Theorem. We are stuck computing an old-fashioned line integral:

$$\int_C \mathbf{Field}(x, y) \bullet \mathbf{unittan}\, ds$$

$$= \int_a^b \mathbf{Field}(x(t), y(t)) \bullet (x'(t), y'(t))\, dt$$

$$= \int_C m(x, y)\, dx + n(x, y)\, dy$$

# Lesson 13: Measuring the Net Flow of a Vector Field ALONG an OPEN Curve in 3-Dimensions

Extending the idea of a line integral (the flow of a 3D vector field along a curve living in 3D) is not particularly difficult. Here it is:

$$\int_C \mathbf{Field(x,y,z)} \bullet \mathbf{unittan\ ds}$$

$$= \int_a^b \mathbf{Field(x(t),y(t),z(t))} \bullet \mathbf{(x'(t),y'(t),z'(t))dt}$$

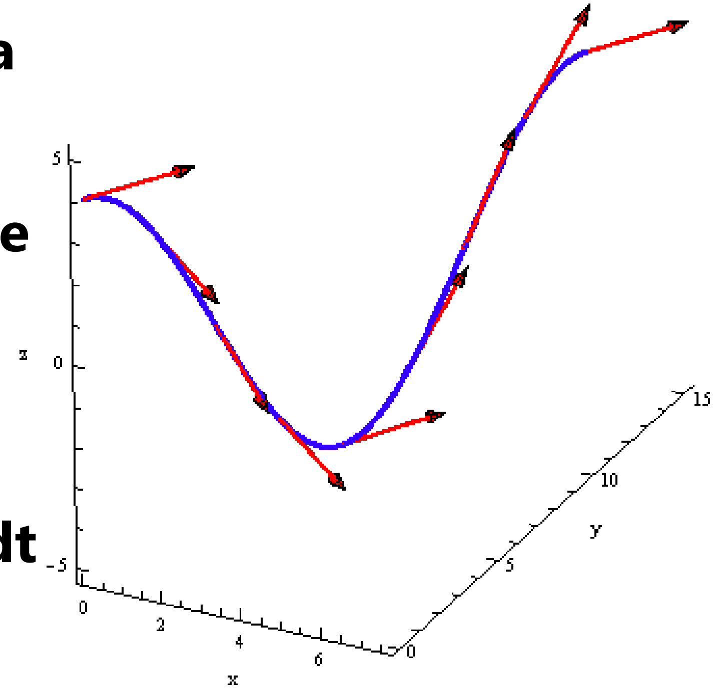
$$= \int_C \mathbf{m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz}$$

**If this integral is positive, the net flow of the vector field along the curve is WITH the direction of the parameterization.**
**If this integral is negative, the net flow of the vector field along the curve is AGAINST the direction of the parameterization.**

**Notice that this is flow along a CURVE, not a surface.**

# Example 1: The Net Flow of a Vector Field ALONG an OPEN Curve in 3-Dimensions

Let $P(t) = \big(x(t), y(t), z(t)\big) = \big(t, 2t, 4\cos(t)\big)$ be a curve in 3D-space for $0 \le t \le 2\pi$. Given $\text{Field}(x,y,z) = \big(-z, x, y\big)$, find the net flow of the vector field along the curve.




$$\int_C \text{Field}(x,y,z) \bullet \text{unittan}\, ds$$

$$= \int_0^{2\pi} \text{Field}(x(t), y(t), z(t)) \bullet (x'(t), y'(t), z'(t))\,dt$$

$$= \int_0^{2\pi} \big(-4\cos(t), t, 2t\big) \bullet \big(1, 2, -4\sin(t)\big)\,dt$$

$$= \int_0^{2\pi} \big(-4\cos(t) + 2t - 8t\sin(t)\big)\,dt$$

$$= 16\pi + 4\pi^2$$

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**Positive. The net flow of the vector field is with the direction of the tangent vectors shown above.**

# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions

Now that we constructed a three-dimensional analog of the net flow of a vector field **ALONG** an **OPEN** curve, we want to generalize the Gauss-Green Theorem so that we can handle the idea of the net flow of a vector field **ALONG** a **CLOSED** curve. This involves significantly more subtlety than you might think at first.

Let's begin by reviewing how we did this in 2-dimensions for a closed curve:

# Lesson 8 : Measuring the Net Flow of a Vector Field ALONG a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ALONG*** the closed curve is measured by:

$$\oint_C \mathbf{Field}(x,y) \bullet \mathbf{unittan}\, ds$$

$$= \int_a^b \mathbf{Field}(x(t),y(t)) \bullet (x'(t),y'(t))dt$$

$$= \oint_C m(x,y)dx + n(x,y)dy$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left( \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dx\, dy$$

$$\text{Let } \mathbf{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}.$$

$$= \iint_R \mathbf{rotField}(x,y)\, dx\, dy$$

# Lesson 8 : The Net Flow of A Vector Field ALONG a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_C \mathbf{Field(x,y)} \bullet \mathbf{unittan}\ ds = \iint_R \mathbf{rotField}\ dx\ dy$$

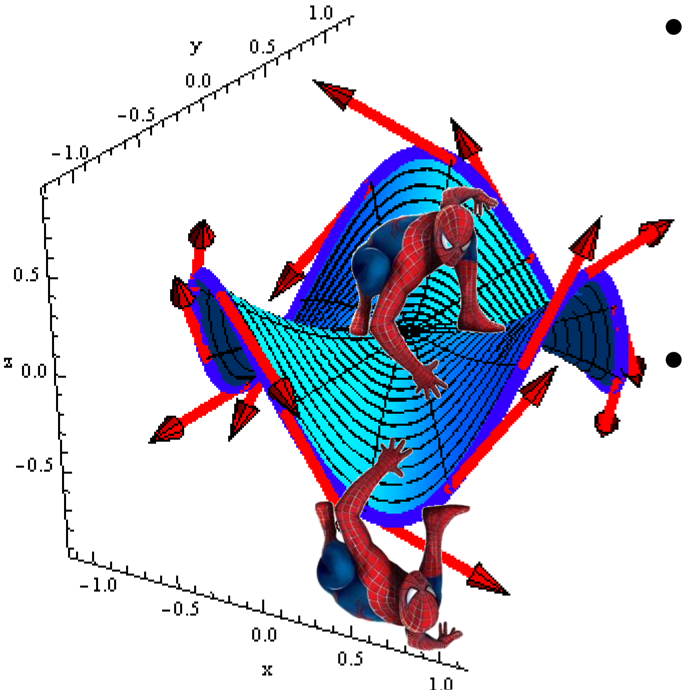
This measures the net flow of the vector field ***ALONG*** the closed curve.

**We define the rotation of the vector field as:**

$$\mathbf{rotField(x,y)} = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = D[n[x,y],x] - D[m[x,y],y]$$

# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

In the 2D case, we needed a counterclockwise parameterization of our curve. The notions of "counterclockwise" and "clockwise" can be problematic in three dimensions, so we need to address this issue!

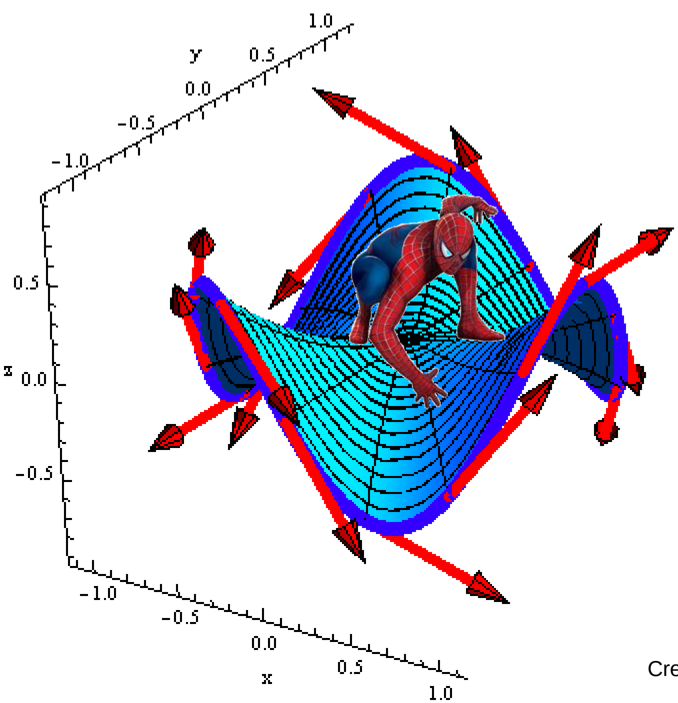



- Imagine you were standing atop this surface. Are your tangent vectors clockwise or counterclockwise relative to you?
- But now image you are standing on the bottom of the surface. Relative to you are the tangent vectors clockwise or counterclockwise?

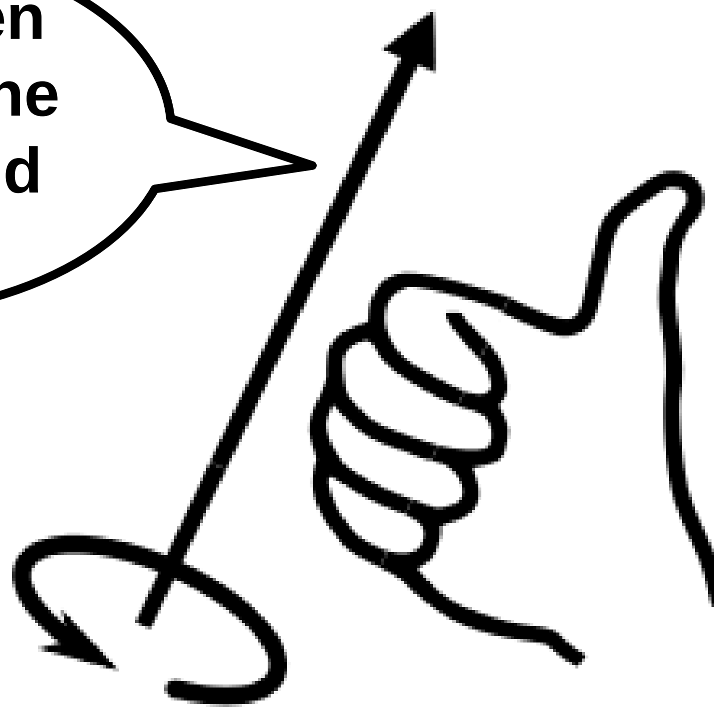
We can sort this problem out in most situations using the "right hand rule":

# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

This gives us our fix. Just designate whatever *you* want to call the top side of the surface, and check that your parameterization is counterclockwise relative to that designation. You may want to pick a "top" that yields a counterclockwise parameterization with minimal work.

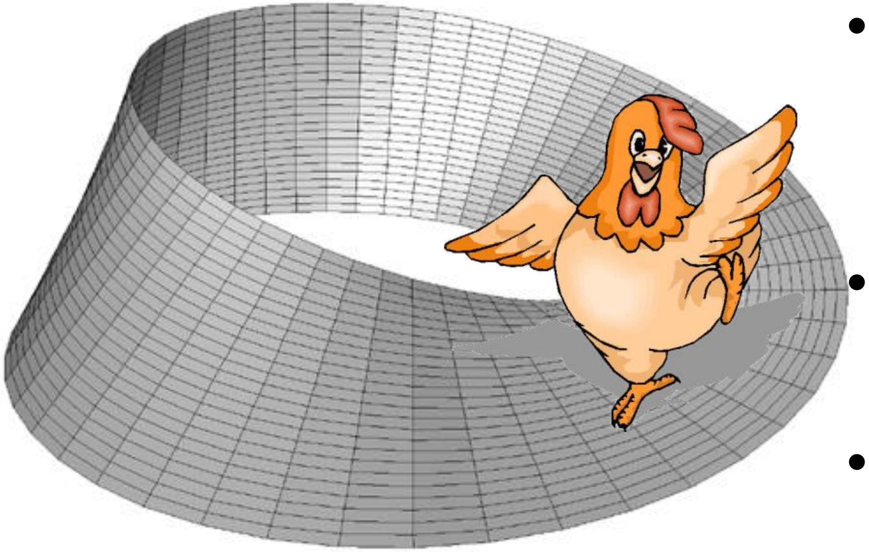



**This is often known as the "Right-hand Rule"**



# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

The vocabulary for a surface that we can designate a top side for is an “orientable” surface. Not all surfaces are orientable, so not all surfaces will be accessible to us using the material in this chapter. See the demo in the Supplemental Mathematica file!



- The Möbius Strip is perhaps the most famous non-orientable surface
- Why did the chicken cross the Möbius Strip?
- To get to the same side!

Next, we are going to need a 3D analog of rotField. This is going to take a little bit of work:

# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

So we've dealt with a few challenges so far:

- How do we define clockwise/counterclockwise in 3D?
  - Answer: Designate a "top" side. Be kind to yourself by picking your "top" to make your boundary curve have a counterclockwise parameterization.
- What if you can't designate a top side?
  - Answer: Then you're in trouble. We need to work with orientable surfaces with 2 distinct sides.
- How do we generalize the rotation of a 2D vector field, rotField(x,y), to 3D?
  - Answer: With a formula in the form of a 3-component vector that can capture the 3 planes of rotation that can occur.

# Lesson 13: The Net Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

What do we mean by 3 planes of rotation? Well, imagine a kayak in whitewater rapids.

Surface rotation

(whirlpool)

xy-rotation

Sideways rotation

(Eskimo roll)

yz-rotation

Forward rotation

(hydraulic/hole)

xz-rotation

# Let's See How we Derived the Formula for "Across" and then Apply these ideas for "Along"

Notice that in Lesson 12, we were able to define divergence as a dot product with del, the differential operator:

Let $F(x,y,z) = \left(m(x,y,z), n(x,y,z), p(x,y,z)\right)$.

Let $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ be known as "del", or the differential operator.

Then $\text{divField}(x,y,z) = \nabla \bullet F = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial p}{\partial z}$.

This allows us to compute the flow of a vector field ACROSS a closed surface.

$$\oiint_S \left(F \bullet n\right) dS = \iiint_V \left(\nabla \bullet F\right) dV$$

We can obtain our higher-dimensional analog to rotField in the same way, but by using a cross product. We will call this curl:

# Lesson 13: The Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

Define the curlField using a cross product:

Let $\mathbf{F}(x,y,z) = \left(m(x,y,z), n(x,y,z), p(x,y,z)\right)$ and let $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ (also known as "del")

$$\text{curlField}(x,y,z) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m & n & p \end{vmatrix}$$

$$= \left(\frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)$$

**This should look like the rotation of the vector field, but redesigned to account for rotation in three different planes!**

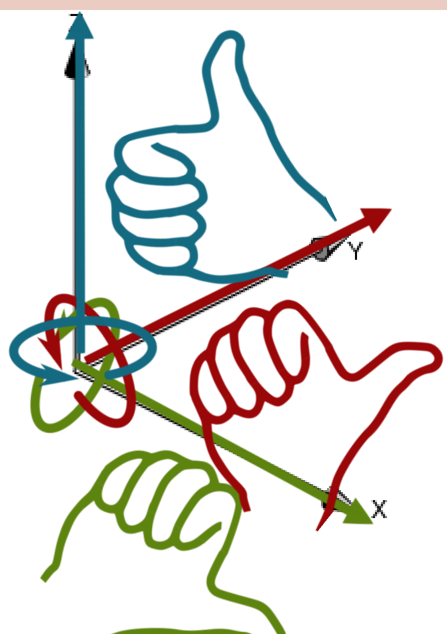
# Lesson 13: The Flow of A Vector Field ALONG a Closed Curve in 3-Dimensions:

$$\mathbf{curlField(x,y,z)} = \left( \frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right)$$

**The x-component of curlField(x,y,z) causes swirl on the yz-plane**

**The y-component of curlField(x,y,z) causes swirl on the xz-plane**

**The z-component of curlField(x,y,z) causes swirl on the xy-plane**



# Making Sense of curlField

Let Field(x,y,z) be a vector field, and let V be a unit vector whose tail is at the point $(x_0,y_0,z_0)$:

**If $\text{curlField}(x_0,y_0,z_0)\bullet V > 0$, then Field(x, y, z) delivers a counterclockwise swirl to the unit vector, V, through the point $(x_0,y_0,z_0)$.**

**If $\text{curlField}(x_0,y_0,z_0)\bullet V < 0$, then Field(x, y, z) delivers a clockwise swirl to the unit vector, V, through the point $(x_0,y_0,z_0)$.**

**If $\text{curlField}(x_0,y_0,z_0)\bullet V = 0$, then Field(x, y, z) delivers no swirl to the unit vector, V, through the point $(x_0,y_0,z_0)$.**

# Example 2: Calculating curlField•V

**Given** $\text{Field}(x,y,z) = \left(-z^2, x^2, y^3\right)$ **and** $V = \dfrac{(2,3,5)}{\sqrt{2^2+3^2+5^2}}$ **with its tail at the point** $(1,0,-1)$, **find and interpret** $\text{curlField}(1,0,-1) \bullet V$:

$$\text{i) curlField}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -z^2 & x^2 & y^3 \end{vmatrix}$$

$$= \mathbf{i}(3y^2 - 0) - \mathbf{j}(0 - (-2z)) + \mathbf{k}(2x - 0)$$

$$= (3y^2, -2z, 2x)$$

$$\text{ii) curlField}(1,0,-1) = (0,2,2)$$

$$\text{iii) curlField}(1,0,-1) \bullet V = (0,2,2) \bullet \frac{(2,3,5)}{\sqrt{2^2+3^2+5^2}} \approx 2.596$$

**Positive, so the vector V feels a counterclockwise swirl around it at the point (1,0, − 1).**

# Stokes' Theorem: Computing the Net Flow of a Vector Field Along a CLOSED Curve in 3D

**Let $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$. Then given an orientable surface, R, with a boundary curve, C, parameterized in the counterclockwise direction such that Field(x,y,z) has no singularities on R, we have:**

$$\oint_C \text{Field}(x,y,z) \bullet \text{unittan}\, ds = \iint_R \text{curlField}(x,y,z) \bullet \text{topunitnormal}\, dA$$

$$\text{Note: curlField}(x,y,z) = \left(\frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)$$

**Note: For this to work, you need your surface to be orientable. That is, you need to be able to designate a top side of the surface, and then get a counterclockwise parameterization of the boundary curve using the right-hand rule as your guide.**

# Stokes' Theorem Using Traditional Notation:

Let $\mathbf{F(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)}$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, and

n = topunitnormal. Then given an orientable surface, R, with a boundary curve, C, with a counterclockwise parameterization such that Field(x,y,z) has no singularities on R, we have:

$$\oint_C \mathbf{F(x,y,z)} \bullet \text{unittan}\, ds = \iint_R \left(\nabla \times \mathbf{F}\right) \bullet \mathbf{n}\, dA$$

$$\text{curlField}(x,y,z) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m & n & p \end{vmatrix} = \left(\frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)$$

# Stokes' 3D Theorem Reduces to Gauss-Green in 2D:

Let $F(x,y) = \big(m(x,y), n(x,y)\big)$ be a 2D-vector field. We can embed it in 3D by writing $F(x,y) = \big(m(x,y), n(x,y), 0\big)$. Hence:

$$\text{curlField}(x,y,z) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m & n & p \end{vmatrix} = \left(0, 0, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) = \big(0, 0, \text{rotField}(x,y)\big)$$

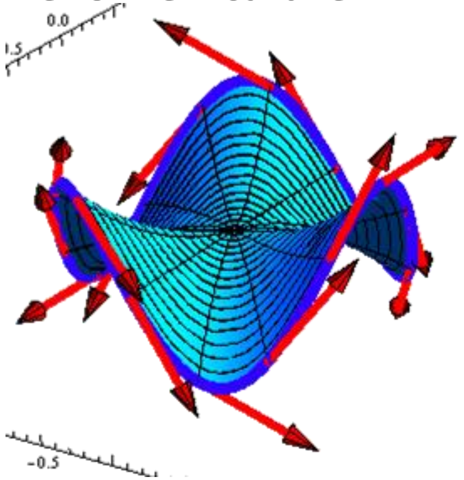
Further, let R be a 2D-region with boundary curve, C, parameterized in the counterclockwise direction. We can embed this in 3D by thinking of this as a flat surface in xyz-space sitting on the xy-plane. Hence, we know that $n = \text{topunitnormal} = (0,0,1)$.

$$\oint_C F(x,y) \bullet \text{unittan}\, ds = \iint_R \big(\nabla \times F\big) \bullet n\, dA$$

$$= \iint_R \text{rotField}(x,y)\, dx\, dy$$

# Example 3: Using Stokes' Theorem

**Let** $\text{Field}(x,y,z) = (y+z, x+z, y+x)$ **and let R be the orientable surface shown below with boundary curve C.**




$$\text{curlField}(x,y,z) = \left(\frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)$$

$$= (0,0,0)$$

$$\oint_C \text{Field}(x,y,z) \bullet \text{unittan } ds = \iint_R \text{curlField} \bullet \text{topunitnormal } dA$$

$$= \iint_R (0,0,0) \bullet \text{topunitnormal } dA$$

$$= 0$$

**So the net flow of the vector field along the curve C is 0.**

# Stokes' Theorem in Practice

**Look at our formula below. If you get a parameterization for your surface, find normal(s,t), and go to the trouble of making sure that these are top normals, it is reasonable to use Stokes' Theorem. In practice, it is usually easier to do the line integral than the surface integral. We will mainly use Stokes' Theorem when the curlField is always (0,0,0).**

$$\oint_C \text{Field}(x,y,z)\bullet\text{unittan}\,ds = \iint_R \text{curlField}\bullet\text{topunitnormal}\,dA$$

$$= \int_{t_1}^{t_2}\int_{s_1}^{s_2} \text{curlField}\big(x(s,t),y(s,t),z(s,t)\big)\bullet\text{normal}(s,t)\,ds\,dt$$

$$\text{normal}(s,t) = \left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right)\times\left(\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right)$$

$$\text{curlField}(x,y,z) = \left(\frac{\partial p}{\partial y}-\frac{\partial n}{\partial z},\frac{\partial m}{\partial z}-\frac{\partial p}{\partial x},\frac{\partial n}{\partial x}-\frac{\partial m}{\partial y}\right)$$

# Lesson 4: The Gradient Vector

**Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables. Then the gradient vector is defined as follows:**

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

**The gradient vector is designed to point in the direction of the greatest INITIAL increase.**

**Notice that the gradient vector always lives in one dimension lower than function does. 3D surface? 2D gradient vector. 2D curve? 1D gradient vector. 4D hypersurface? 3D gradient vector.**

# The Gradient Vector Through the Lens of "Del," The Differential Operator

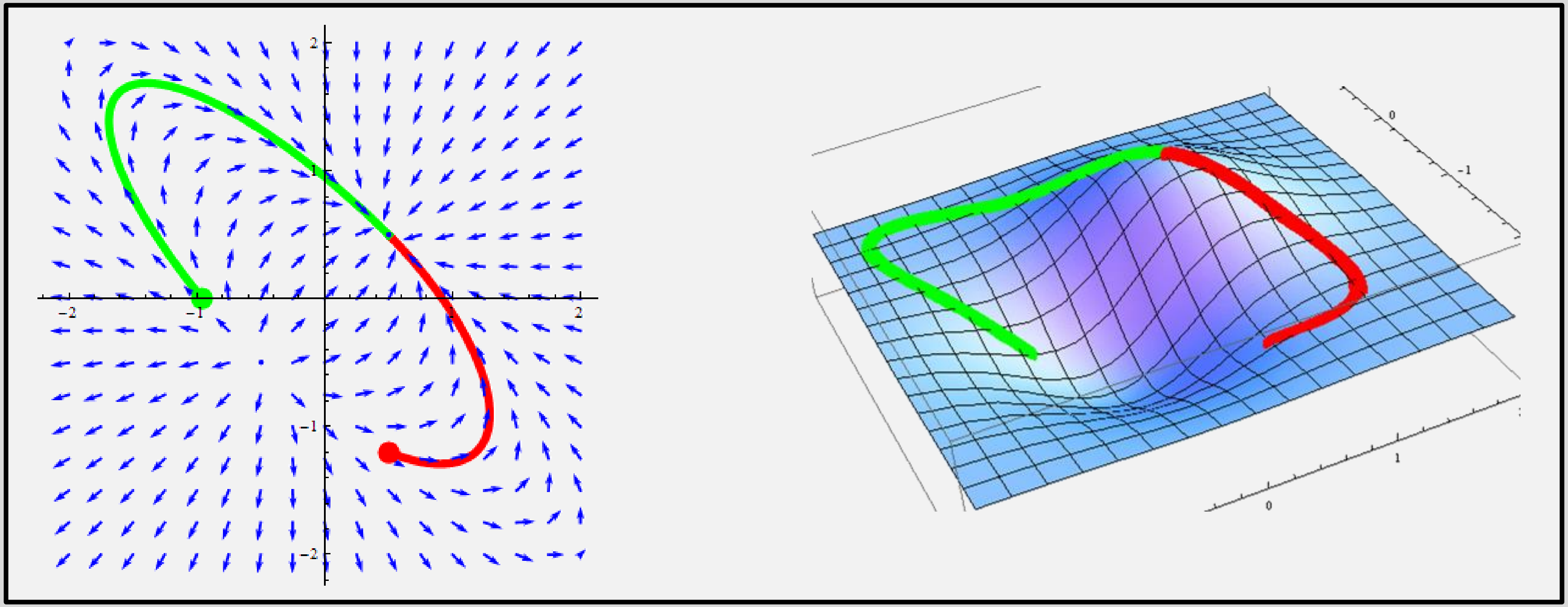
**Let $f(x_1, x_2, \dots, x_n)$ be a function of n variables and let $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$.**

$$\nabla f = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) f$$

$$= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$$

# Lesson 5: The Gradient Points Towards the Direction of Greatest Initial Increase

**Compare the gradient field and $f(x, y) = \dfrac{x + y}{e^{x^2 + y^2}}$ :**

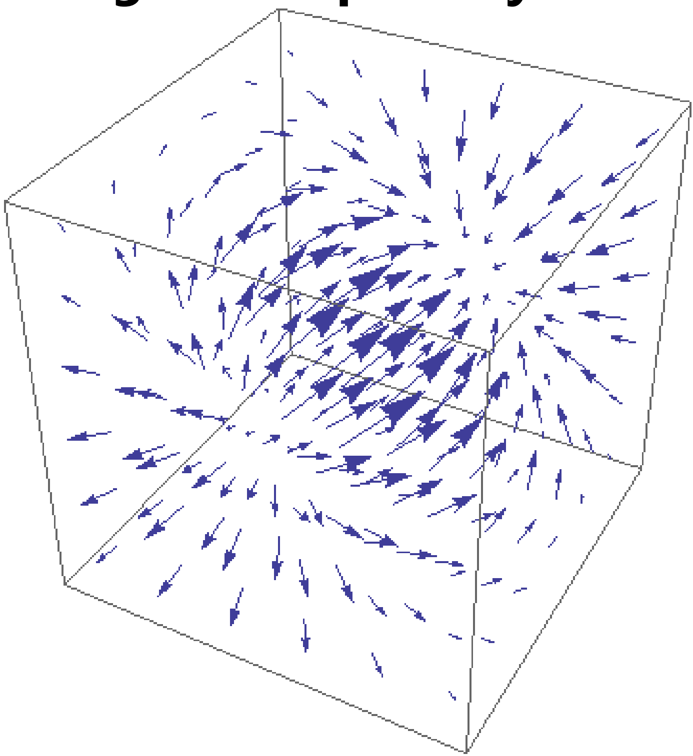



**Following the gradient (usually) gets us to local mins/maxes.**

# Lesson 13: The Gradient Points Towards the Direction of Greatest Initial Increase

**Now consider $f(x,y,z) = \dfrac{x+y+z}{e^{x^2+y^2+z^2}}$. Now, $\nabla f(x,y,z)$ is a 3-dimensional vector field and $f(x,y,z)$ is a 4-dimensional hypersurface. Your best bet is to think of $f(x,y,z)$ as a temperature function with local maximums being little hot spots and local minimums being little cold spots. The gradient points you to these locations:**

**We can use FindMaximum in Mathematica to find a hot spot at $(0.408, 0.408, 0.408)$. You can also see this from the plot of the vector field.**



# Lesson 7: The 2D Gradient Test

**A vector field, $\text{Field}(x,y) = \big(m(x,y), n(x,y)\big)$, is a gradient field if and only if the vector field has no singularities and:**

$$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$$

**Equivalently: $\text{rotField}(x,y) = 0$**

## Proof of "if" Part of Theorem:

**If $\text{Field}(x,y)$ is a gradient field, then $\text{Field}(x,y) = \big(f_x, f_y\big)$.**

**So $f_{xy} = f_{yx}$.**

# Lesson 13: The 3D Gradient Test

A vector field, $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$, is a gradient field if and only if the vector field has no singularities and:

$$\text{curlField}(x,y,z) = (0,0,0)$$

$$\text{curlField}(x,y,z) = \left(\frac{\partial p}{\partial y} - \frac{\partial n}{\partial z}, \frac{\partial m}{\partial z} - \frac{\partial p}{\partial x}, \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)$$

Outline of "if" Direction of Proof: For a function f(x,y,z), we would have to show that the gradient field, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, has $\text{curlField} = (0,0,0)$.

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (0,0,0)$$

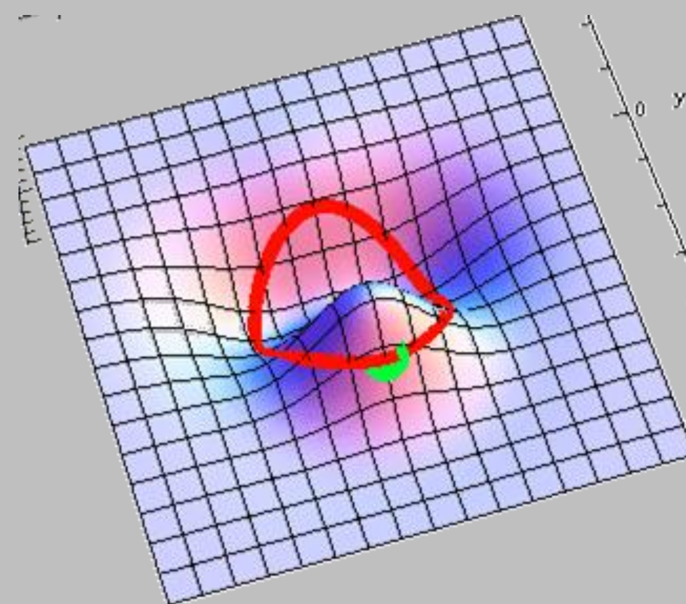
# Lesson 7: The Net Flow of a Gradient Field Along a Closed Curve

**Let Field(x,y)=$\big(m(x,y), n(x,y)\big)$ be a gradient field, and let C be a simple closed curve with a parameterization (x(t),y(t)) for $a \le t \le b$.**

1) $\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))\,dt = 0$

2) $\oint_C m(x,y)\,dx + n(x,y)\,dy = 0$

3) **The net flow of a gradient field along a simple closed curve is 0.**



**Is the flow of a gradient field ACROSS a closed curve 0?**

# Lesson 13: The Net Flow of a Gradient Field Along a Closed Curve in 3D

Let $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$ be a gradient field, and let C be a simple closed curve with a parameterization $(x(t), y(t), z(t))$ for $a \le t \le b$.

1) $\int_a^b \text{Field}(x(t), y(t), z(t)) \bullet (x'(t), y'(t), z'(t))\,dt = 0$

2) $\oint_C m(x,y,z)\,dx + n(x,y,z)\,dy + p(x,y,z)\,dz = 0$

3) The net flow of a gradient field along a simple closed curve in 3D is 0.

<u>Proof:</u> If $\text{Field}(x,y,z)$ is a gradient field, then $\text{curlField}(x,y,z) = (0,0,0)$.

Stokes' Theorem: $\oint_C \text{Field}(x,y,z) \bullet \text{unittan}\,ds = \iint_R \text{curlField} \bullet \text{topunitnormal}\,dA$

$$= \iint_R (0,0,0) \bullet \text{topunitnormal}\,dA$$

$$= 0$$

# Lesson 13: The Net Flow of a Gradient Field Along a Closed Curve in 3D

Let $\text{Field}(x,y,z)=\big(m(x,y),n(x,y),p(x,y)\big)$ be a gradient field, and let C be a simple closed curve with a parameterization $(x(t),y(t),z(t))$ for $a \le t \le b$.

1) $\int_a^b \text{Field}(x(t),y(t),z(t)) \bullet (x'(t),y'(t),z'(t))dt = 0$

2) $\oint_C m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz = 0$

3) The net flow of a gradient field along a simple closed curve in 3D is 0.

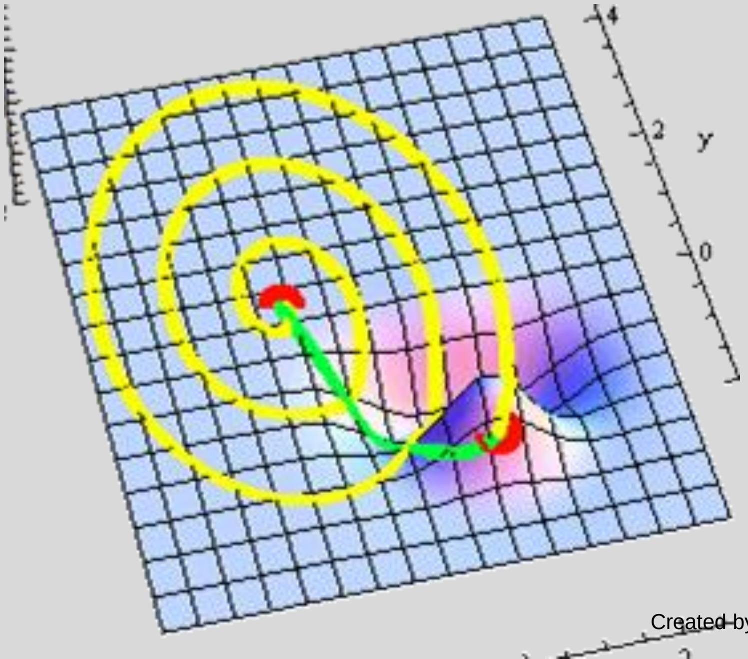
**Your intuition here should be that if you start and end the same point, your net change in temperature is 0.**

**Note that this DOES NOT work for any old vector field. The key is that you must have a GRADIENT field.**

# Lesson 7: Path Independence for 2D-Gradient Field

**Let Field(x,y) = $\left(m(x,y), n(x,y)\right)$ be a gradient field, and let $C_1$ and $C_2$ be different curves that share the same starting and ending point:**

$$\int_{C_1} m(x,y)dx + n(x,y)dy = \int_{C_2} m(x,y)dx + n(x,y)dy$$




**A gradient field is said to be <u>path independent</u>. The net flow of the gradient field along any two curves connecting the same two points is the same...**

# Lesson 13: Path Independence for a 3D-Gradient Field

Let $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$ be a gradient field, and let $C_1$ and $C_2$ be different curves that share the same starting and ending point:

$$\int_{C_1} m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz = \int_{C_2} m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz$$

**Proof:** Since $\text{Field}(x,y,z)$ is a gradient field, the net flow of the vector field along a closed curve is 0:

Let C be the closed curve formed by $C = C_1 \cup C_2$.

Then $\oint_C \text{Field}(x,y,z) \bullet \text{unittan ds} = 0.$

$$\Rightarrow \int_{C_1} \text{Field}(x,y,z) \bullet \text{unittan ds} - \int_{C_2} \text{Field}(x,y,z) \bullet \text{unittan ds} = 0$$

$$\Rightarrow \int_{C_1} \text{Field}(x,y,z) \bullet \text{unittan ds} = \int_{C_2} \text{Field}(x,y,z) \bullet \text{unittan ds}$$

# Lesson 13: Path Independence for a 3D-Gradient Field

Let $\text{Field}(x,y,z) = \big(m(x,y,z), n(x,y,z), p(x,y,z)\big)$ be a gradient field, and let $C_1$ and $C_2$ be different curves that share the same starting and ending point:

$$\int_{C_1} m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz = \int_{C_2} m(x,y,z)dx + n(x,y,z)dy + p(x,y,z)dz$$

**A gradient field is said to be path independent.**

**The net flow of the gradient field along any two curves that start and end at the same points is the same…**