Lesson 1

Plotting Circles and Ellipses Parametrically

Example 1: The Unit Circle

Let's compare traditional and parametric equations for the unit circle :

Traditional :

$$x^2 + y^2 = 1$$

Parametric :

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$$

$$P(t) = \left(\cos(t), \sin(t)\right)$$

t is called a parameter $\rightarrow$ $0 \le t < 2\pi$

You can see that the parametric equation satisfies the traditional equation by substituting one into the other:

$$\begin{aligned} x^2 + y^2 &= 1 \\ (\cos(t))^2 + (\sin(t))^2 &= 1 \\ 1 &= 1 \end{aligned}$$

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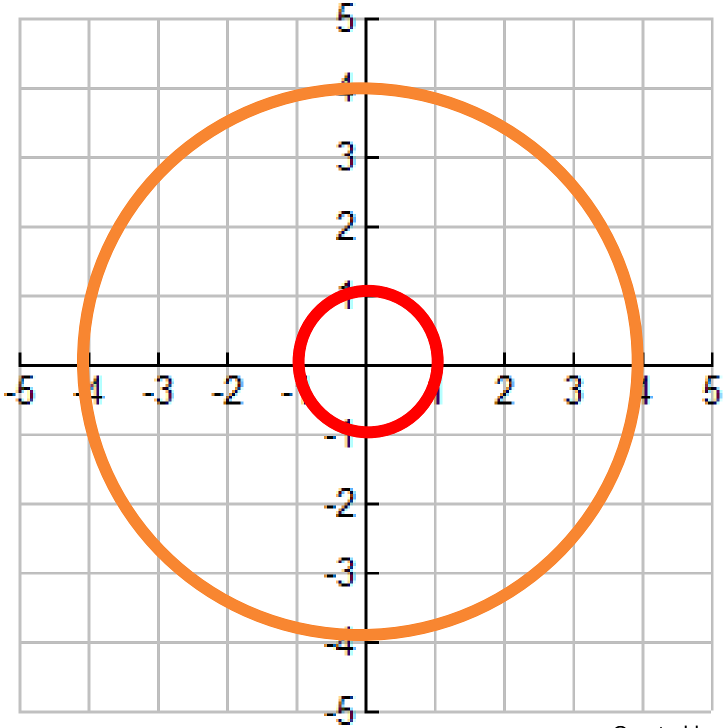
Example 2: Circle of Radius r

Let's compare traditional and parametric equations for a circle of radius r centered at the origin:

Traditional:

$$x^2 + y^2 = r^2$$



Parametric:

$$\begin{cases} x(t) = r\cos(t) \\ y(t) = r\sin(t) \end{cases}$$

$$P(t) = r\left(\cos(t), \sin(t)\right)$$

$$0 \le t < 2\pi$$

Orientation: Counterclockwise

Think of this as dilating the unit circle by a factor of r.

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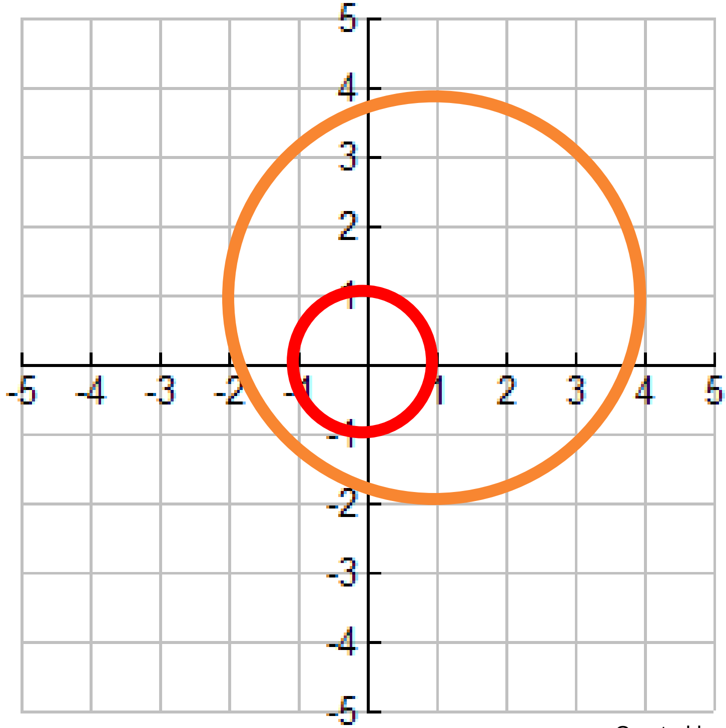
Example 3: Recentering the Circle

Let's compare traditional and parametric equations for a circle of radius r centered at (h, k) :

Traditional :

$$(x-h)^2+(y-k)^2=r^2$$



Parametric :

$$\begin{cases} x(t)=r\cos(t)+h \\ y(t)=r\sin(t)+k \end{cases}$$

$$P(t)=r\left(\cos(t),\sin(t)\right)+(h,k)$$

$$0\leq t<2\pi$$

Think of this as dilating the unit circle by a factor of r and translating by the point (h,k). Let's add the orientation:

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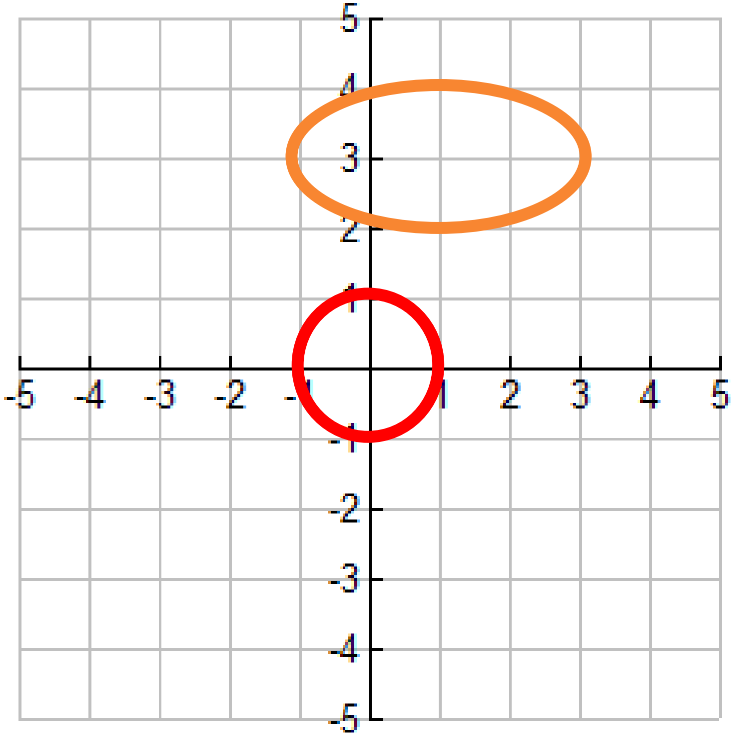
Example 4: Ellipses

Let's compare traditional and parametric equations for an ellipse centered at (h,k) :

Traditional :

$$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$$



Parametric :

$$\begin{cases} x(t)=a\cos(t)+h \\ y(t)=b\sin(t)+k \end{cases}$$

$$P(t)=\big(a\cos(t),b\sin(t)\big)+(h,k)$$

$$0\le t<2\pi$$

Think of this as dilating the unit circle by a factor of "a" in the x-direction, a factor of "b" in the y-direction, and translated by the point (h,k). Now add orientation!

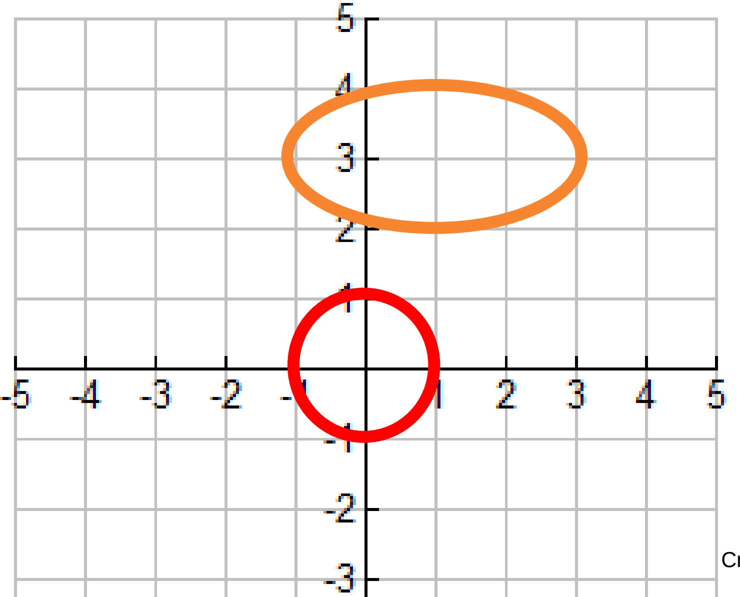
Example 5: Find the Equation

Find the traditional and parametric equation for the orange ellipse.

Traditional :

$$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$$

$$\left(\frac{x-1}{2}\right)^2+\left(y-3\right)^2=1$$



Parametric :

$$P(t)=\left(a\cos(t),b\sin(t)\right)+(h,k)$$

$$P(t)=\left(2\cos(t),\sin(t)\right)+(1,3)$$

$$(0\le t<2\pi)$$

$$\begin{cases}x(t)=2\cos(t)+1\\ y(t)=\sin(t)+3\end{cases}$$

Orientation : Counterclockwise

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Example 6: Find the Equation

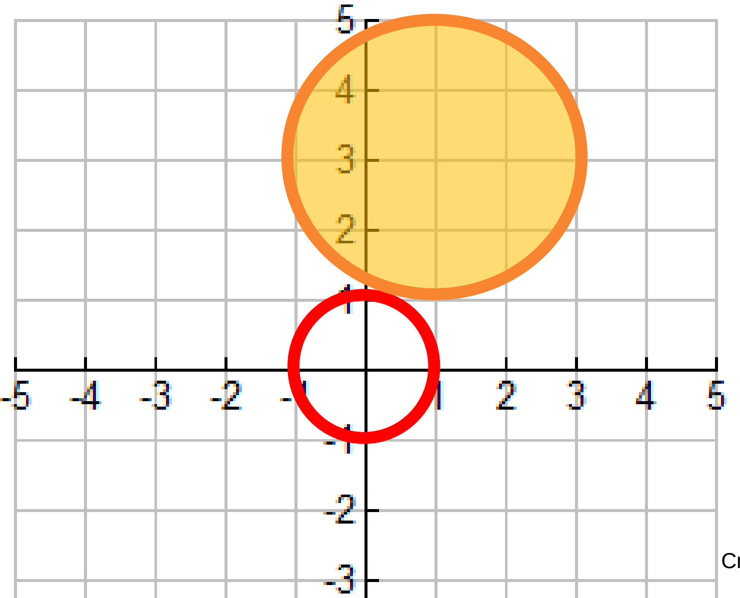
Find the parametric equation for the disc.

Boundary Curve : $P(t) = 2\left(\cos(t), \sin(t)\right) + (1,3)$

$$0 \le t < 2\pi$$

Whole Disc : $P(r,t) = r\left(\cos(t), \sin(t)\right) + (1,3)$

$$0 \le t < 2\pi,\ 0 \le r \le 2$$



We added a second parameter to get this surface:
The first parameter, t, controlled the fact that we had a full rotation of a circle.
The second parameter, r, controlled the fact that we have a filled-in disc of radius r=2.

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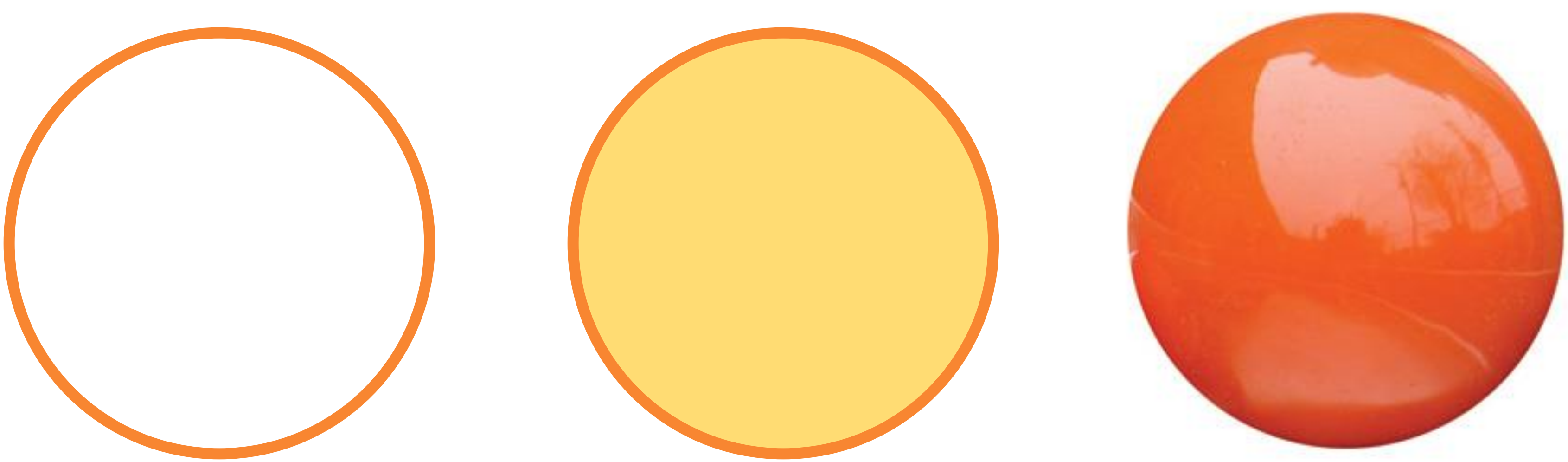
Curves / Surfaces / Solids

<u>Curve :</u> Only one parameter, you can trace it out with the tip of your finger... P(t)

<u>Surface :</u> You need two parameters, you would need to trace it out with the palm of your hand... P(r,t)

<u>Solid :</u> You need three parameters, you would need to fill it with water or sand... P(r,s,t)



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Example 7: Two Ways of Saying the Same Thing

Show that the following parametrically-defined curves are equivalent:

$$P(t) = 3\left(\cos(t), \sin(t)\right) + (5, 2)$$

$$P(t) = \left(3\cos(t), 3\sin(t)\right) + (5, 2)$$

$$P(t) = \left(3\cos(t) + 5, 3\sin(t) + 2\right)$$

$$\begin{cases} x(t) = 3\cos(t) + 5 \\ y(t) = 3\sin(t) + 2 \end{cases}$$

and

$$\begin{cases} x(t) = 3\cos(t) + 5 \\ y(t) = 3\sin(t) + 2 \end{cases}$$

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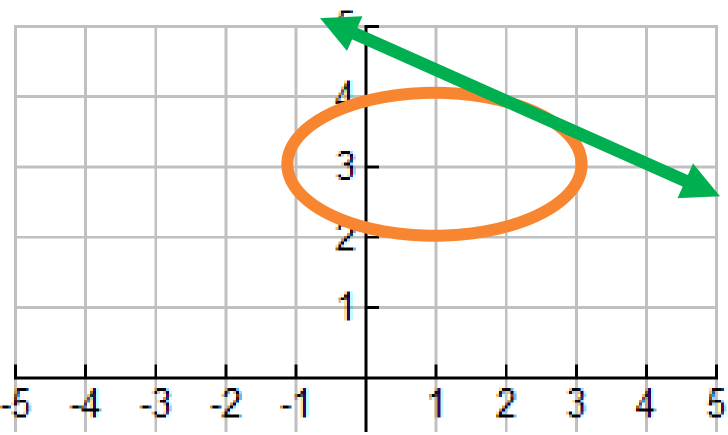
Example 8: Parametric Calculus

Find the right-most and bottom-most points on the orange ellipse.

Find the equation of the green tangent line shown at x=2.

$$\begin{cases} x(t) = 2\cos(t) + 1 \\ y(t) = \sin(t) + 3 \end{cases}$$

$$\begin{cases} x'(t) = -2\sin(t) \\ y'(t) = \cos(t) \end{cases}$$



$$x'(t) = -2\sin(t) = 0$$

$$t = 0, \pi$$

$$(x(0), y(0)) = (3,3)$$

$$(x(\pi), y(\pi)) = (-1,3)$$

$$y'(t) = \cos(t) = 0$$

$$t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = (1,4)$$

$$\left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) = (1,2)$$

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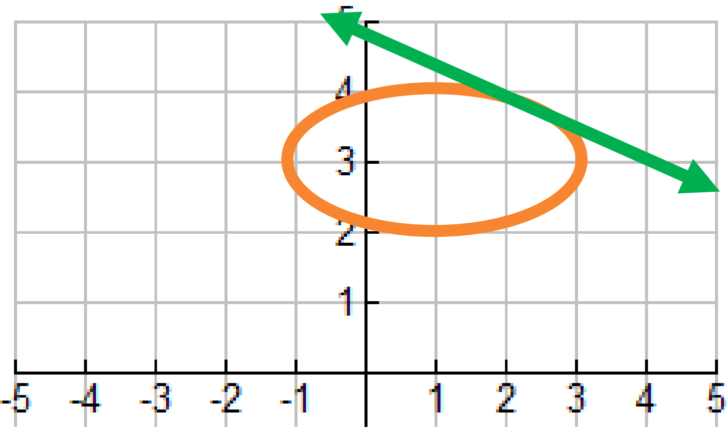
Example 8: Parametric Calculus

Find the right-most and bottom-most points on the orange ellipse.

Find the equation of the green tangent line shown at x=2.

$$\begin{cases} x(t) = 2\cos(t) + 1 \\ y(t) = \sin(t) + 3 \end{cases}$$

$$\begin{cases} x'(t) = -2\sin(t) \\ y'(t) = \cos(t) \end{cases}$$



$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos(t)}{-2\sin(t)} = -\frac{1}{2}\cot(t)$$

$$x(t) = 2\cos(t) + 1$$
$$2 = 2\cos(t) + 1$$
$$1 = 2\cos(t)$$
$$\cos(t) = \frac{1}{2}$$
$$t = \frac{\pi}{3}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = -\frac{1}{2}\cot\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{6}$$

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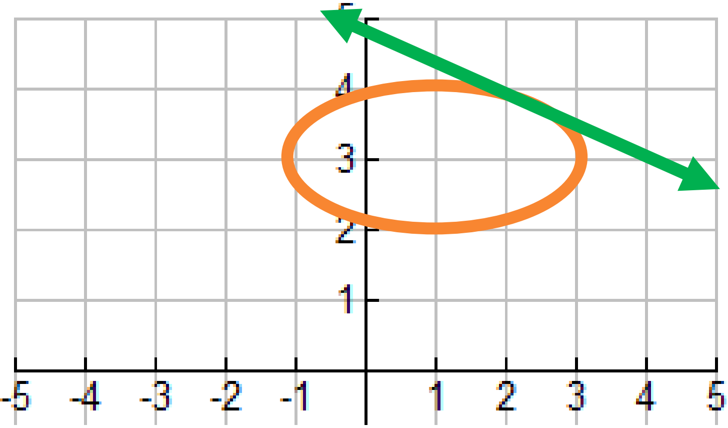
Example 8: Parametric Calculus

Find the right-most and bottom-most points on the orange ellipse.

Find the equation of the green tangent line shown at x=2.

$$\begin{cases} x(t) = 2\cos(t) + 1 \\ y(t) = \sin(t) + 3 \end{cases}$$



$$\begin{cases} x\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right) + 1 = 2 \\ y\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + 3 = 3 + \frac{\sqrt{3}}{2} \end{cases}$$

$$m = -\frac{1}{2}\cot\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{6}$$

$$y - 3 - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{6}(x - 2)$$

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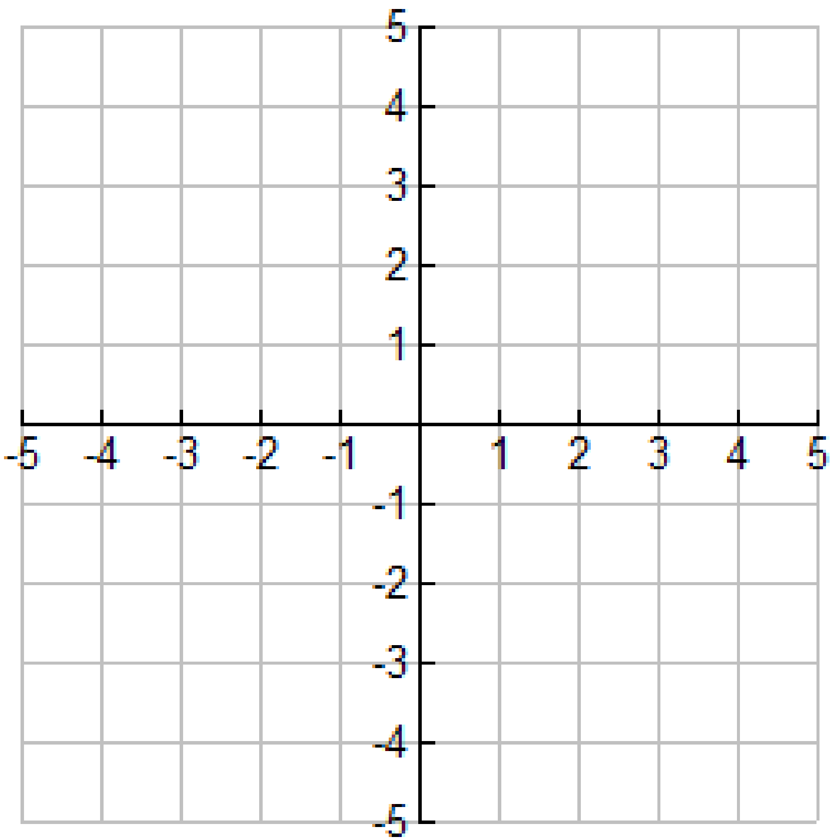
Example 9: Don't Assume!

Plot $\left(2\sin(t), 3\cos(t)\right)$ for $0 \le t \le \frac{3\pi}{2}$.

Include orientation.



t	x	y

Don't just assume this is a normal ellipse! If something looks "off," make a table!!!!!

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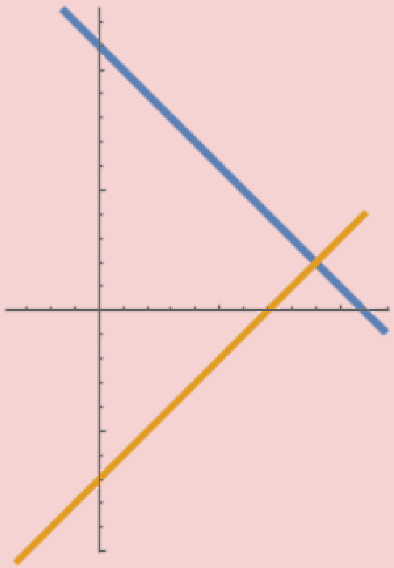
Example 10: Intersections of Parametric Curves

Where do the following curves intersect?

$$L_1(t) = (5 - t, t + 6) \text{ and } L_2(t) = (t + 3, t - 4)$$

Step 1: Plot



Step 2: Solve for "s" and "t"

$L_1(t) = L_2(s)$

$(5 - t, t + 6) = (s + 3, s - 4)$

$5 - t = s + 3$ and $t + 6 = s - 4$

$s + t = 2$ and $s - t = 10$

Solve this system :

$s = 6$ and $t = -4$

Step 3: Find intersection

$L_1(-4) = (5 + 4, 6 - 4)$

$= (9, 2)$

$L_2(6) = (6 + 3, 6 - 4)$

$= (9, 2)$

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Example 10: Warning

Where do the following curves intersect?

$$L_1(t) = (5 - t, t + 6) \text{ and } L_2(t) = (t + 3, t - 4)$$

Why can't we solve $L_1(t) = L_2(t)$ **instead?**

Compare this code to find out :

```
L1[t_]={5 - t, t + 6};
L2[t_]={t + 3, t – 4};
Solve[L1[t]==L2[t],t]
```

```
L1[t_]={5 - t, t + 6};
L2[t_]={t + 3, t – 4};
Solve[L1[t]==L2[s],{s,t}]
```

Example 11: Entering the Third Dimension

Let $P(s,t) = (2\cos(t), 2\sin(t), s)$ for $0 \le t < 2\pi$ and $1 \le s \le 4$.

Notice there are two parameters...

This must be a 2D surface living in 3D!

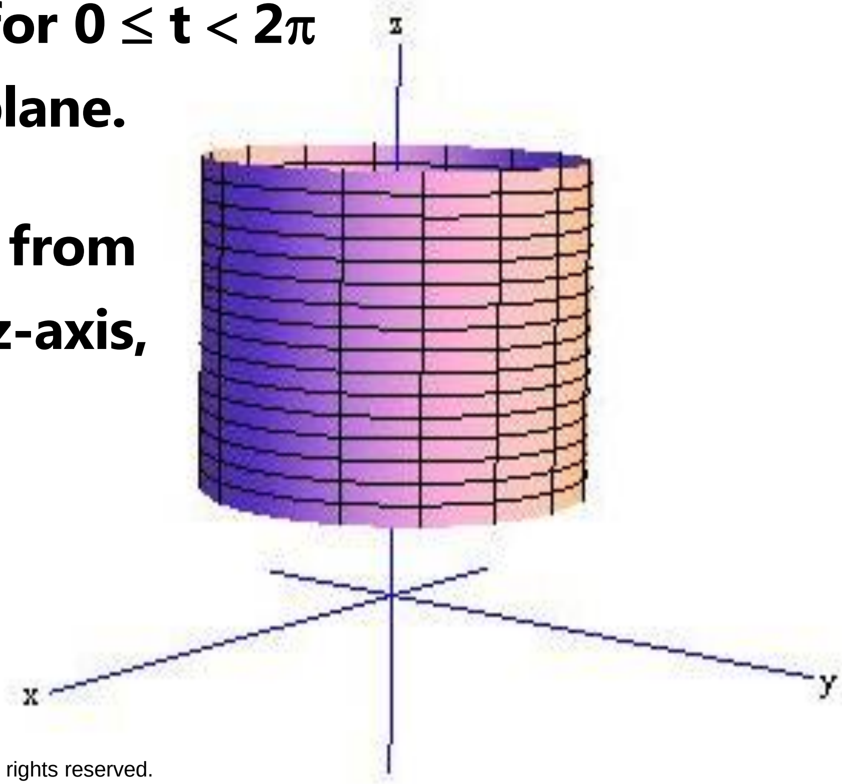
Equivalent form:

$P(s,t) = (2\cos(t), 2\sin(t), 0) + (0,0,s)$

The $x(s,t) = 2\cos(t)$ and $y(s,t) = 2\sin(t)$ for $0 \le t < 2\pi$ gives you a circle of radius 2 on the xy-plane.

$z(s,t) = s$ for $1 \le s \le 4$ drags those circles from a height of 1 up to a height of 4 on the z-axis, making a cylinder!

This is a cylinder of height 3 with a base of radius 2 centered at (0,0,1)!



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Try it Yourself in Mathematica (Copy and Paste!)

```
Clear[a, b, s, t, u, v, ThreeAxes]
ThreeAxes[u_, v_] :=
  Graphics3D[{{Blue, Line[{{-u, 0, 0}, {u, 0, 0}}]},
    Text["x", {u + v, 0, 0}], {Blue, Line[{{0, -u, 0}, {0, u, 0}}]},
    Text["y", {0, u + v, 0}], {Blue, Line[{{0, 0, 0}, {0, 0, u}}]},
    Text["z", {0, 0, u + v}]}];
ThreeAxes[u_] := ThreeAxes[u, u/8];
Manipulate[
 Show[ThreeAxes[5, .2],
  ParametricPlot3D[{2 Cos[a], 2 Sin[a], b}, {a, 0, .01 + t}, {b,
    1, .01 + s}, PlotRange -> {{-3, 3}, {-3, 3}, {0, 5}},
   AxesLabel -> {"x", "y", "z"}], ViewPoint -> {3, 3, 1},
  Boxed -> False], {t, 0, 2*Pi}, {s, 1, 4}]
```

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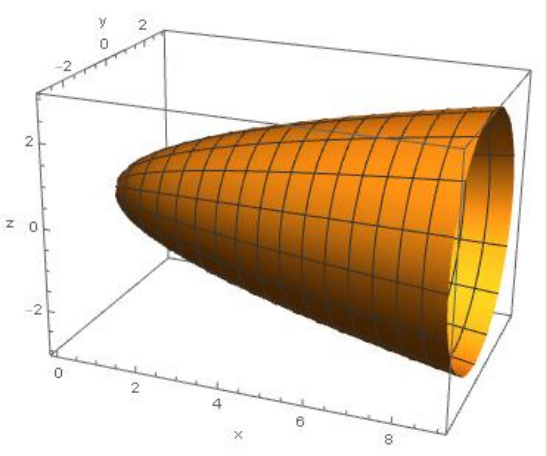
Example 12: Surface of Revolution

Plot $y = \sqrt{x}$ revolved around the x-axis for $0 \le x \le 9$.

Step 1: Sketch a picture



Step 2: Describe the Surface in Words

This curve has circular cross sections parallel to the yz-plane with centers at (x,0,0) and radii of $\sqrt{x}$.

Step 3: Translate into an Equation

$$P(x,t) = \left(x,0,0\right) + \sqrt{x}\left(0,\cos(t),\sin(t)\right)$$

$$0 \le t < 2\pi,\ 0 \le x \le 9$$

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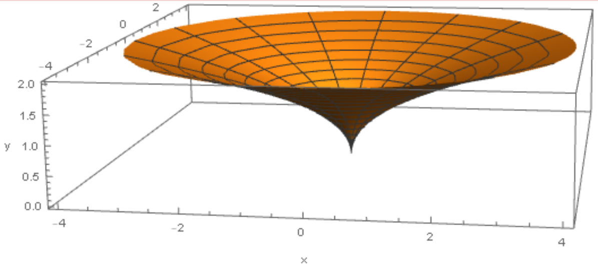
Example 12: Surface of Revolution

Plot $y = \sqrt{x}$ revolved around the y-axis for $0 \le y \le 2$.

Step 1: Sketch a picture



Step 2: Describe the Surface in Words

This curve has circular cross sections parallel to the xz-plane with centers at (0,y,0) and radii of y^2. This is because $y = \sqrt{x}$ turns into $x = y^2$.

Alternatively, you can substitute in $y = \sqrt{x}$ to get circles with centers at $(0,\sqrt{x},0)$ and radii of x.

Step 3: Translate into an Equation

$$P(y,t) = \left(0, y, 0\right) + y^2\left(\cos(t), 0, \sin(t)\right)$$

$$0 \le t < 2\pi,\ 0 \le y \le 2$$

$$P(x,t) = \left(0, \sqrt{x}, 0\right) + x\left(\cos(t), 0, \sin(t)\right)$$

$$0 \le t < 2\pi,\ 0 \le x \le 4$$

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Example 12: Summary

1. Sketch a plot of the 3D surface by hand. Be sure to label your axes.
2. Find an expression describing the centers of your circular cross sections. Let's call it **center(x)**.
3. Find an expression describing the radius of your circular cross sections in terms of the same variable as #2. Let's call it **radius(x)**
4. Determine whether your cross sections are perpendicular to the xy-plane, yz-plane, or xz-plane.
   - For xy-plane, use **cross(s) = (cos(s), sin(s), 0)**.
   - For yz-plane, use **cross(s) = (0,cos(s), sin(s)**.
   - For xz-plane, use **cross(s) = (cos(s), 0, sin(s)**.
5. Write your final answer:

   surface(s,x) = **center(x)** + **radius(x)** **cross(s)**