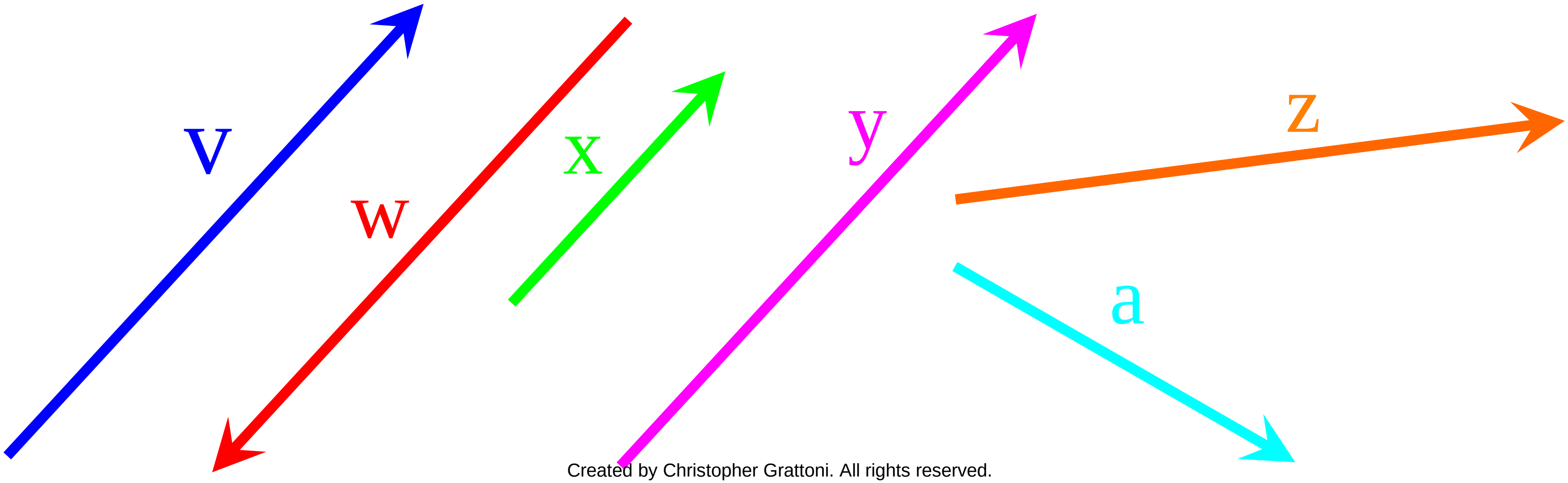
Lesson 2

Vectors

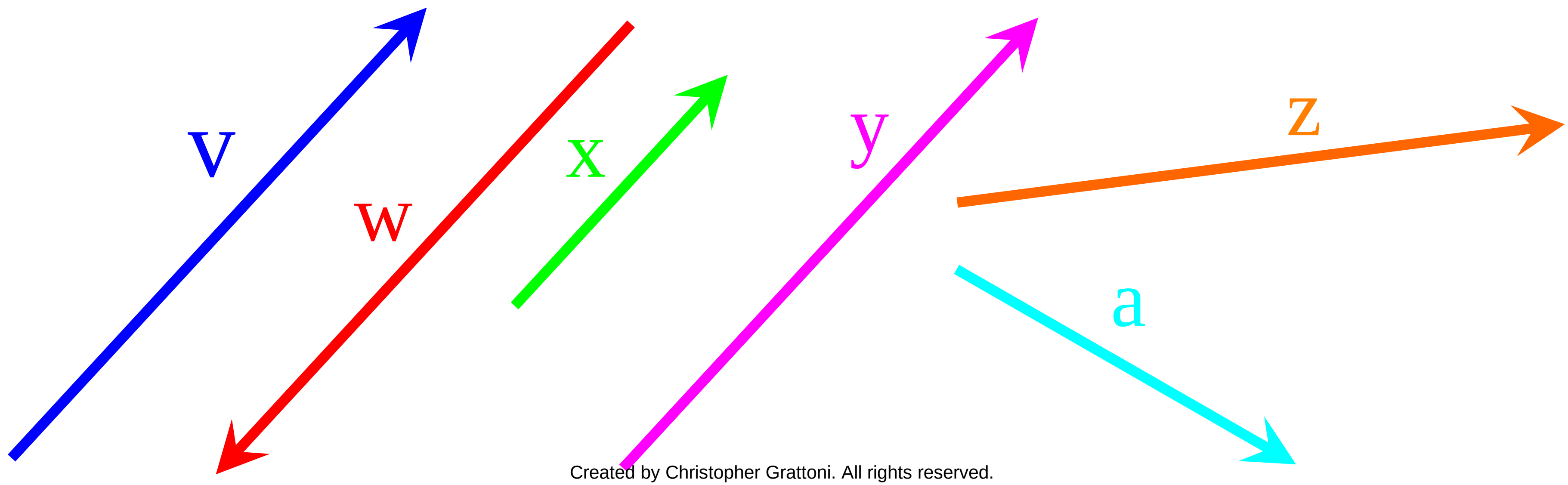
Definition of a Vector

- A vector is a quantity that has both a magnitude and a direction
- Vectors encode more information than scalars (magnitude without direction)
- Vectors can be represented in the plane as a directed line segment (magnitude represented by length, direction represented by arrowhead)



Definition of a Vector

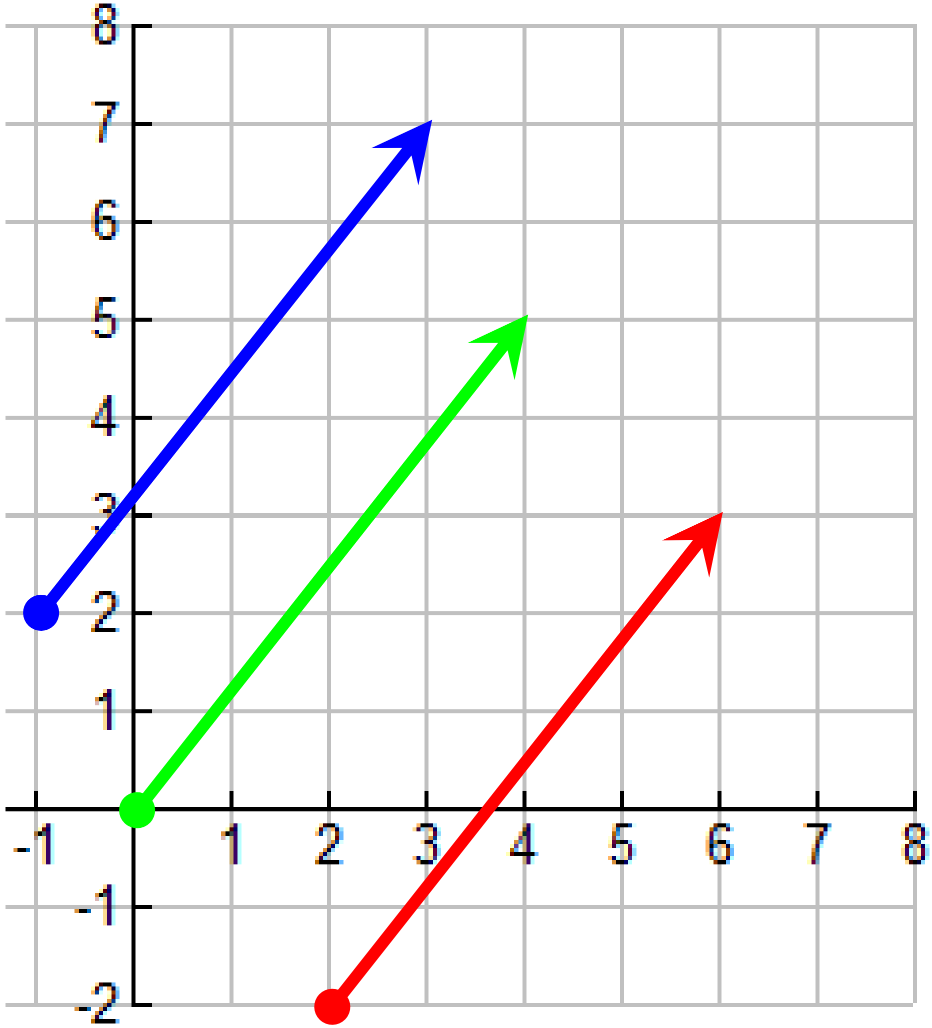
- Two vectors are equal if and only if they have the same magnitude and direction, but they don't have to start from the same point
- Only "v" and "y" are equal in the list below
- A useful way to think of a vector is as a force such as a gust of wind
- A vector is NOT a line segment, it is NOT a ray, and it is not a locus of points… keep trying to think of it as a direction and a magnitude (a force or a push)



Example 1: Defining a Vector By Its Tip and Tail

Describe each of the vectors shown on the right.

These are all the same vector, just with different tails/tips.



$$\text{Tip} - \text{Tail} = \text{Vector}$$

$$(4,5) - (0,0) = (4,5)$$

$$(3,7) - (-1,2) = (4,5)$$

$$(6,3) - (2,-2) = (4,5)$$

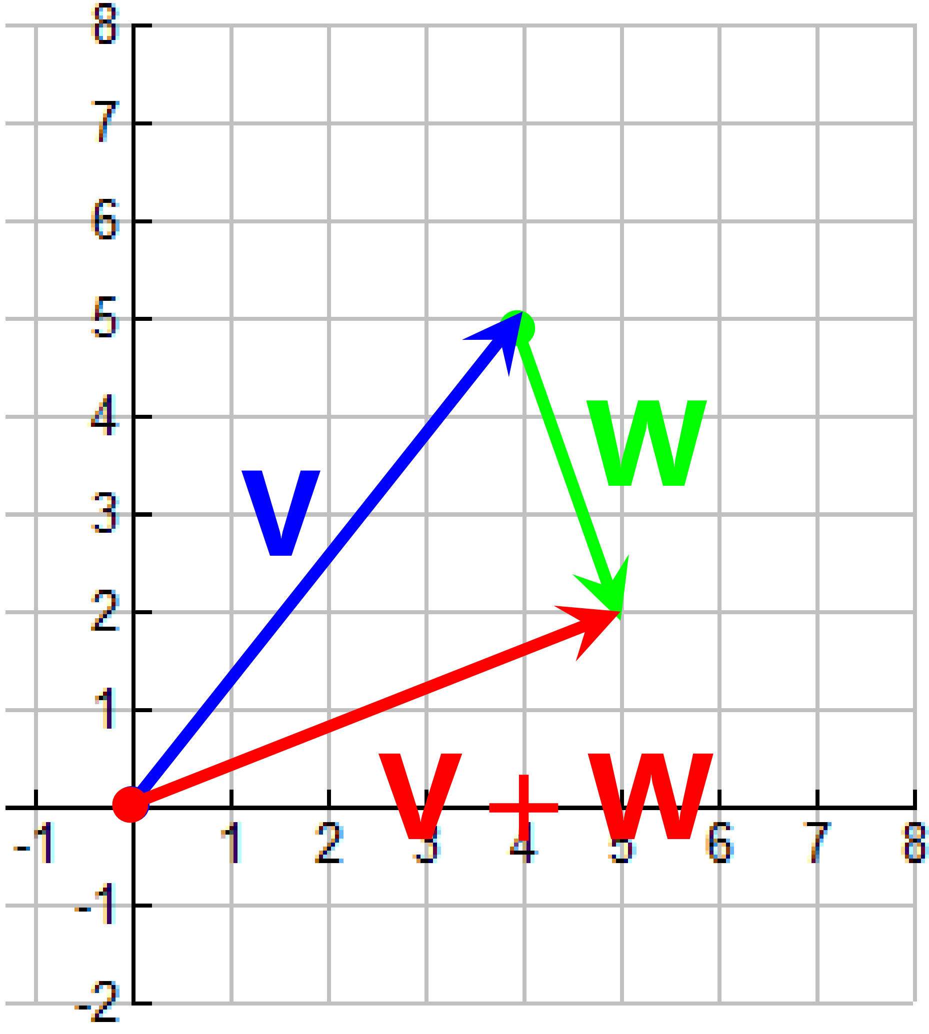
Example 2: V + W and V – W

Let V = (4,5) and W = (1,–3).

Find the following :

i) V + W :

$$(4,5) + (1,-3) = (5,2)$$



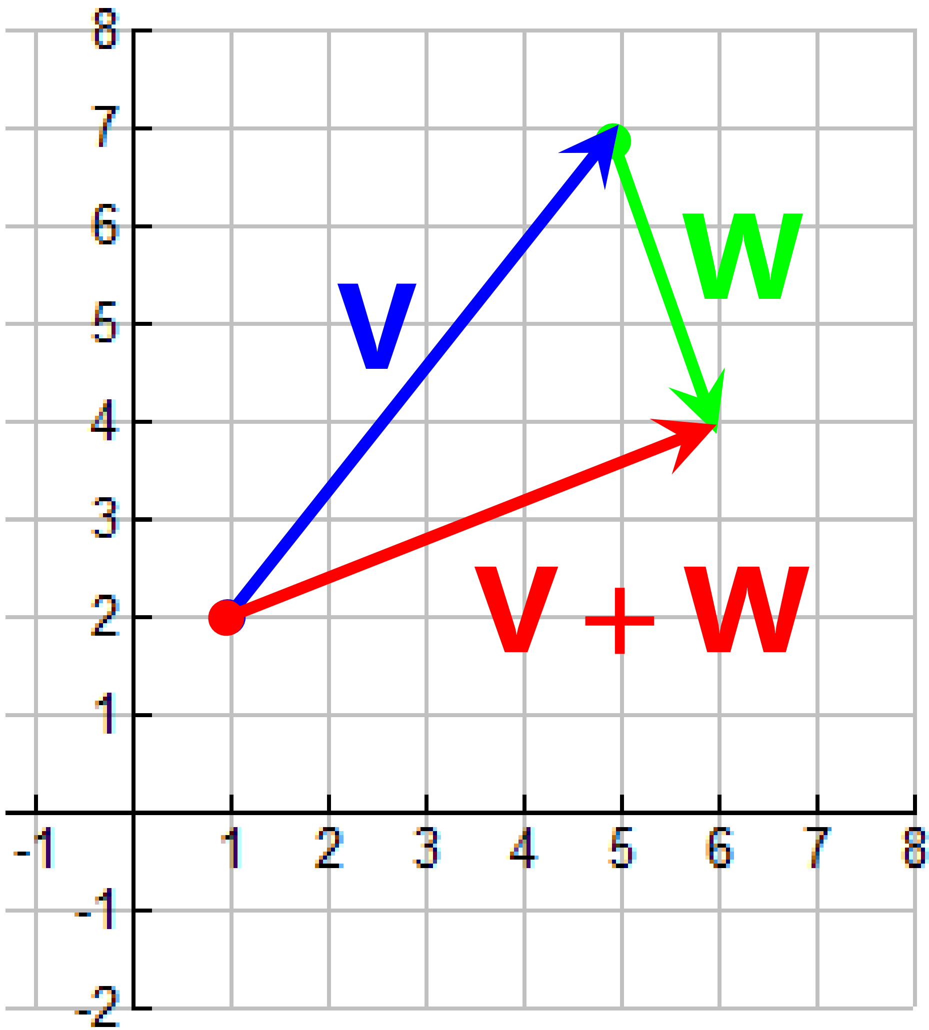
Example 2: V + W and V – W

Let V = (4,5) and W = (1,–3).

Find the following :

i) V + W :

$$(4,5) + (1,-3) = (5,2)$$



Example 2: V + W and V – W

Let **V = (4,5)** and **W = (1,–3)**.

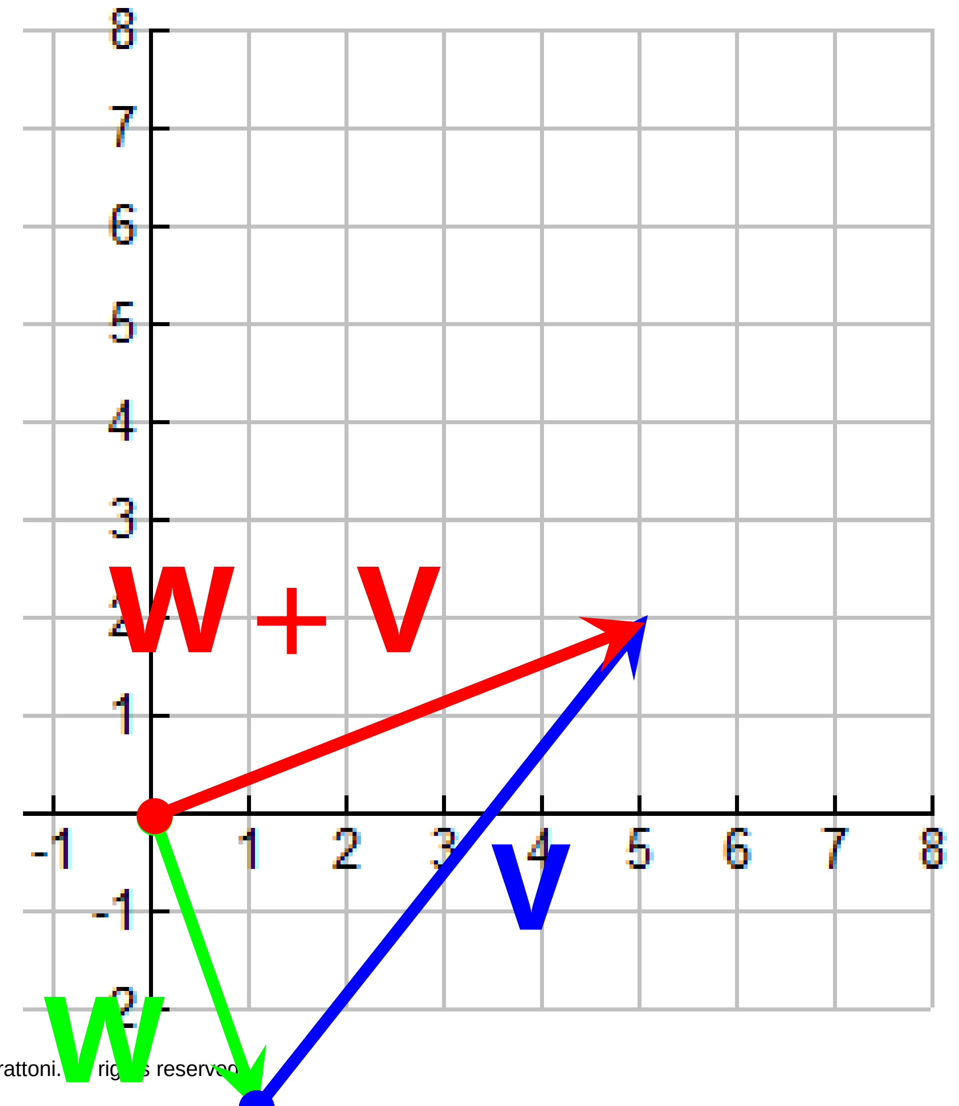
Find the following :

i) V + W :

$$(4,5)+(1,-3)=(5,2)$$

ii) W + V :

$$(1,-3)+(4,5)=(5,2)$$



Example 2: V + W and V – W

Let **V** = (4,5) and **W** = (1,–3).

Find the following :

i) V + W :

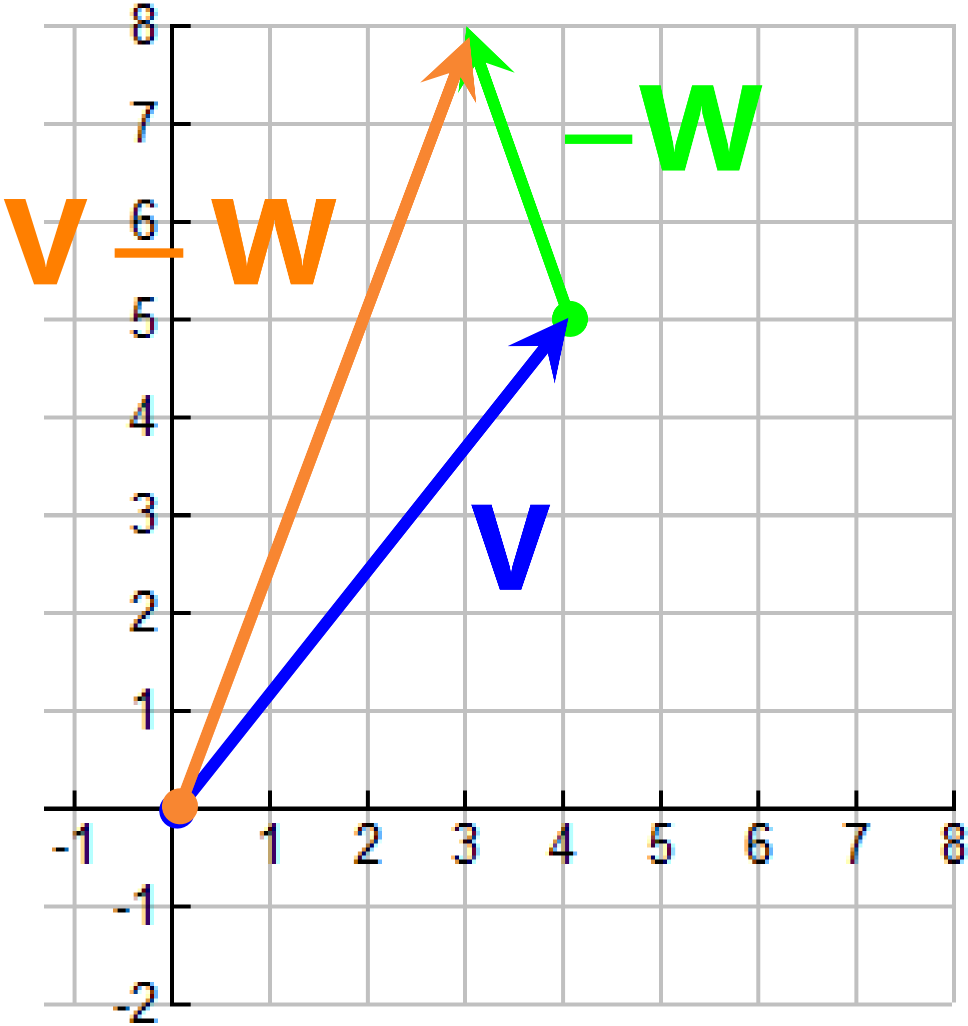
$(4,5)+(1,-3)=(5,2)$

ii) W + V :

$(1,-3)+(4,5)=(5,2)$

iii) V – W = V + (–W) :

$(4,5)-(1,-3)=(3,8)$



Example 2: V + W and V – W

Let $V = (4,5)$ and $W = (1,-3)$.

Find the following :

i) $V + W$:

$$(4,5) + (1,-3) = (5,2)$$

ii) $W + V$:

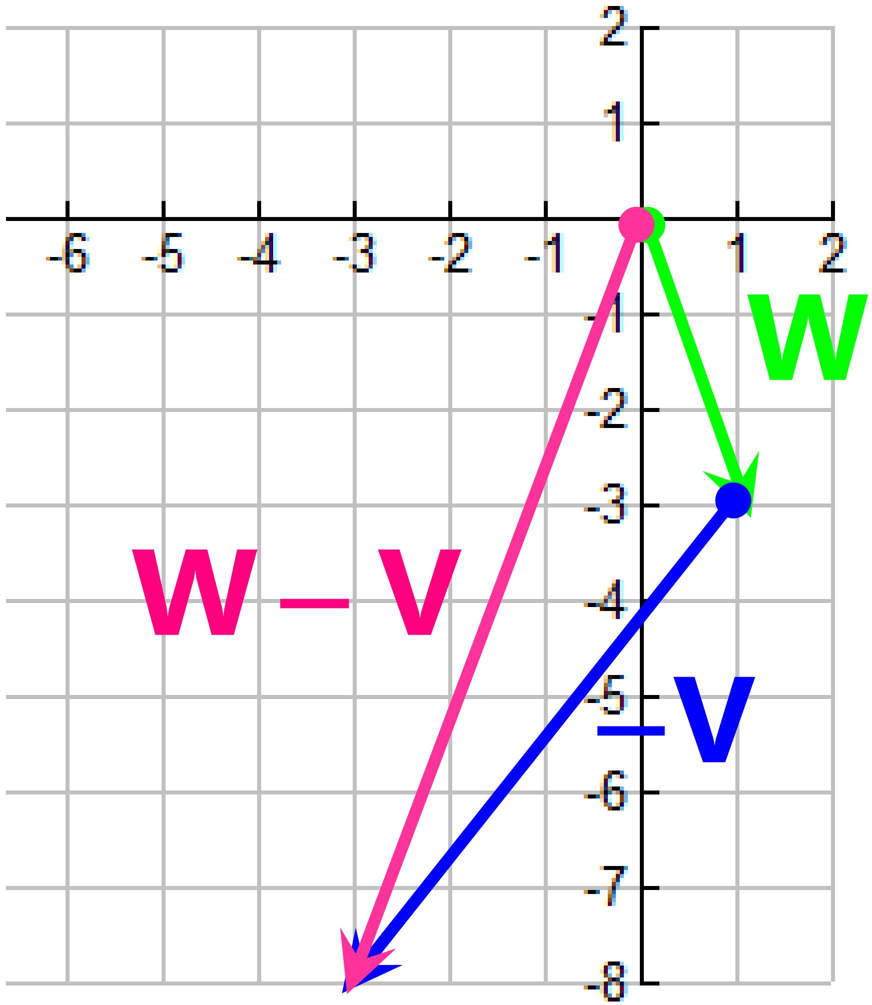
$$(1,-3) + (4,5) = (5,2)$$

iii) $V - W = V + (-W)$:

$$(4,5) - (1,-3) = (3,8)$$

iv) $W - V$:

$$(1,-3) - (4,5) = (-3,-8)$$



Example 3: Dot Product

2D : $V \bullet W = (v_1, v_2) \bullet (w_1, w_2) = v_1 w_1 + v_2 w_2$

3D : $V \bullet W = (v_1, v_2, v_3) \bullet (w_1, w_2, w_3) = v_1 w_1 + v_2 w_2 + v_3 w_3$

Length of Vector : $|V| = \sqrt{V \bullet V}$

Distance Between Two Points, P and Q : $\sqrt{(P - Q) \bullet (P - Q)}$

Alternative Dot Product Formula : $V \bullet W = |V||W| \cos(\theta)$

(θ is the angle between V and W)

Example 3: Dot Product

Find the angle between V and W :

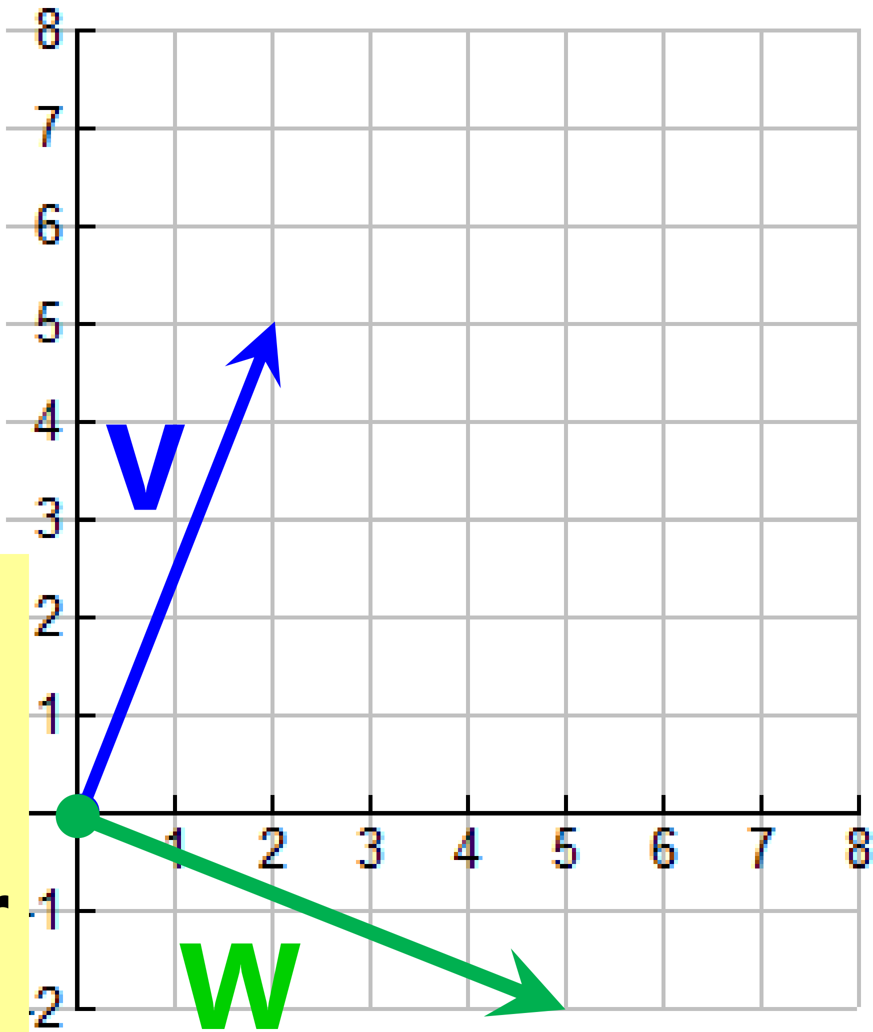
$$\mathbf{V} \bullet \mathbf{W} = (2,5) \bullet (5,-2) = 0$$

$$0 = |\mathbf{V}||\mathbf{W}|\cos(\theta) \Rightarrow \theta = 90^\circ$$

They are perpendicular!

Theorem : If $\mathbf{V} \bullet \mathbf{W} = 0$, then $\mathbf{V} \perp \mathbf{W}$.

(As long as neither V nor W is a zero vector)



Example 4: Projection/Push

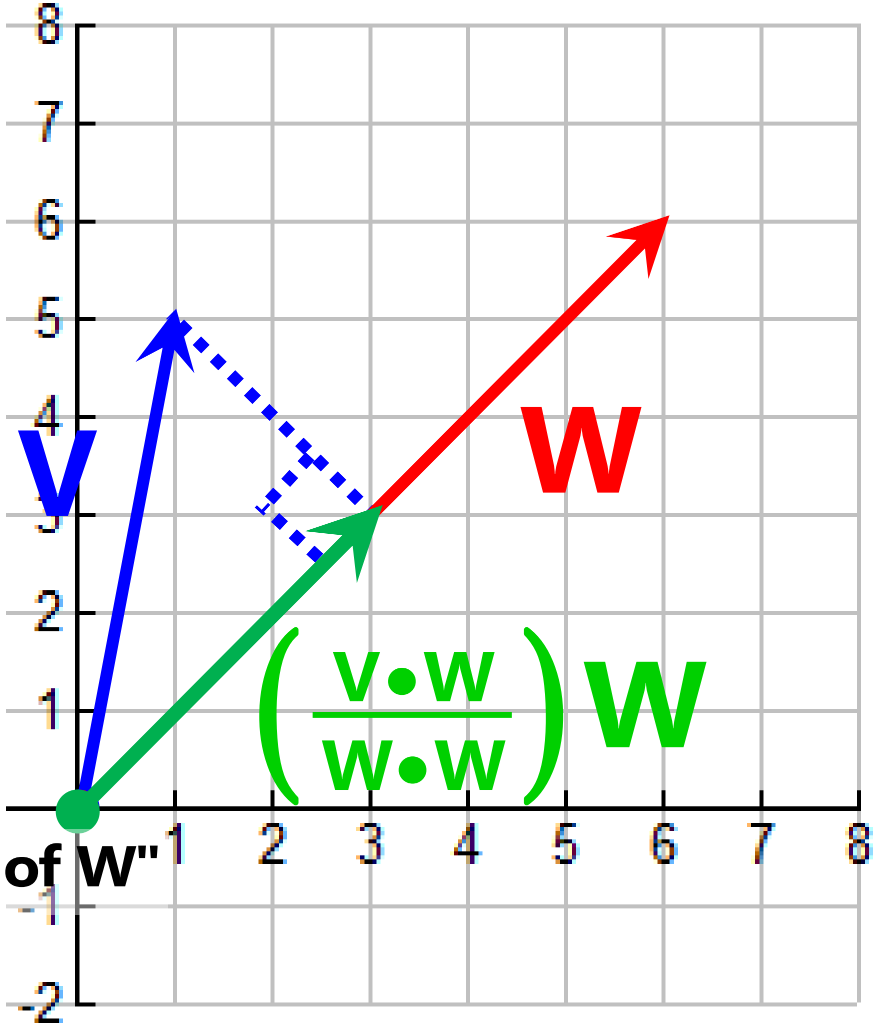
Let **V** = (1,5) and **W** = (6,6).

Find the following:

i) $\left(\dfrac{V \bullet W}{W \bullet W}\right) W$:

$$\left(\frac{(1)(6)+(5)(6)}{(6)(6)+(6)(6)}\right)(6,6) = \frac{1}{2}(6,6)$$

$$= (3,3)$$



ii) Interpret:

"The projection of V onto W"

"The push of V in the direction of W"

"The component of V in the direction of W"

Is the push of V with or against W?

Example 5: Projection/Push

Let $\color{blue}{V=(1,5)}$ and $\color{red}{W=(2,2)}$.

Find the following:

i) $\left(\dfrac{V\bullet W}{W\bullet W}\right)\color{red}{W}$:

$$\left(\frac{(1)(2)+(5)(2)}{(2)(2)+(2)(2)}\right)(2,2)=\frac{3}{2}(2,2)$$

$$=(3,3)$$

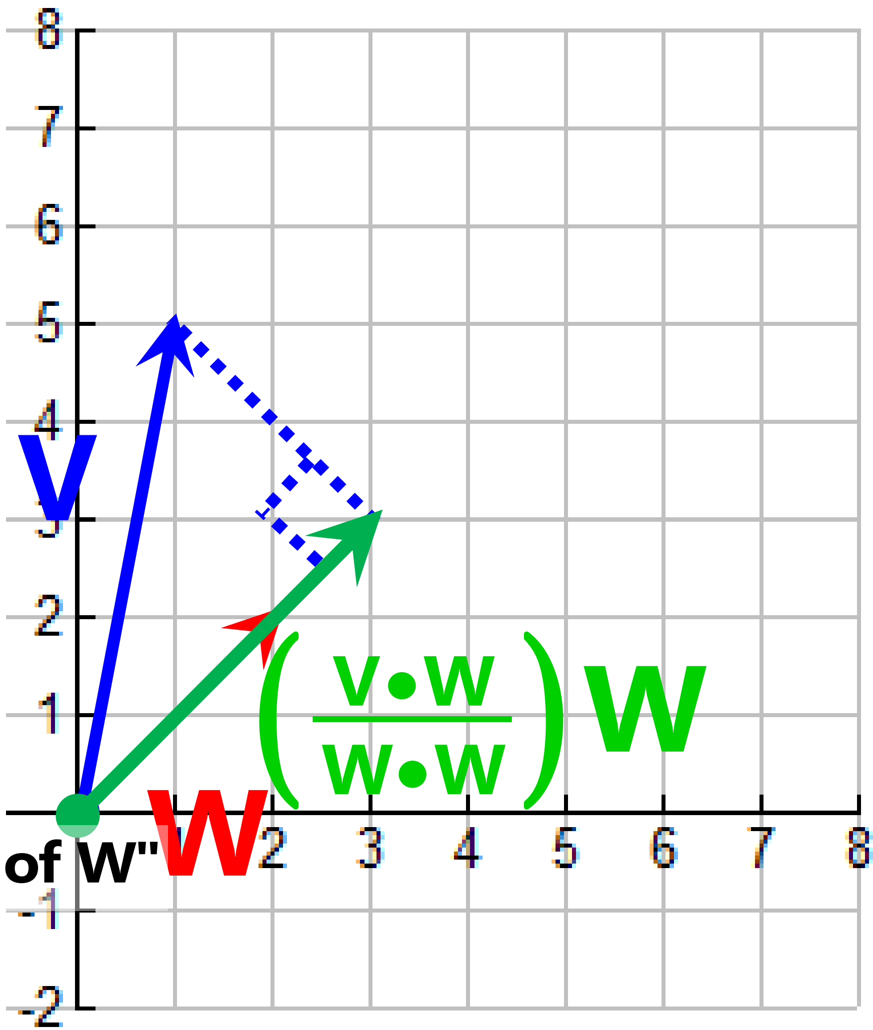
ii) Interpret:

"The projection of V onto W"

"The push of V in the direction of W"

"The component of V in the direction of W"

Is the push of V with or against W?



Example 6: Projection/Push

Let $\mathbf{V} = (1,5)$ and $\mathbf{W} = (-1,-1)$.

Find the following:

i) $\left(\dfrac{\mathbf{V}\bullet\mathbf{W}}{\mathbf{W}\bullet\mathbf{W}}\right)\mathbf{W}$:

$$\left(\frac{(1)(-1)+(5)(-1)}{(-1)(-1)+(-1)(-1)}\right)(-1,-1)$$

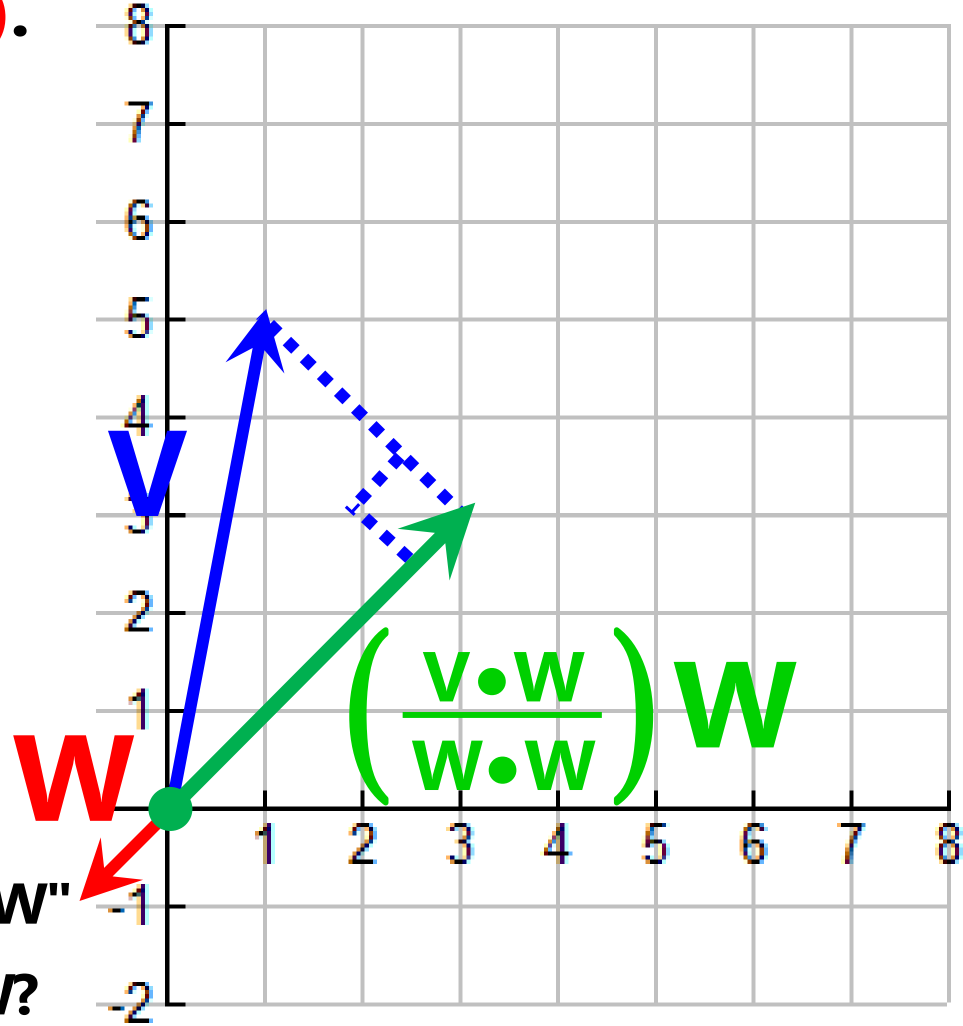
$$= -\frac{6}{2}(-1,-1)$$

$$= (3,3)$$

ii) Interpret:

"The push of V in the direction of W"

Is the push of V with or against W?



Example 7: Defining a Line Parametrically

a) Find the equation of the line shown at the right.

Slope of Line?

$$\frac{5}{4}$$

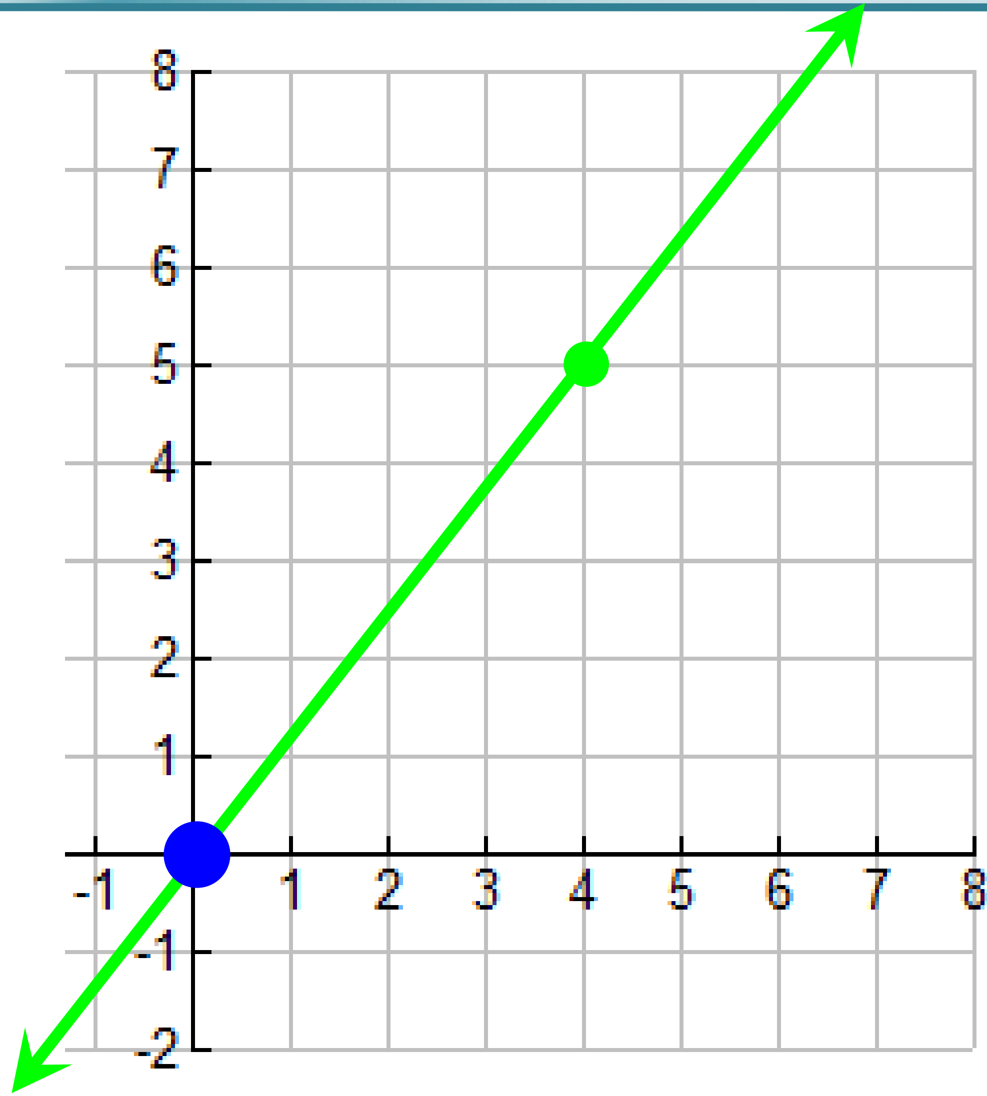
Vector Through Given Points?

$$(4,5)-(0,0)=(4,5)$$

Equation of Line:

$$L(t)=(0,0)+t(4,5)$$

$$t\in(-\infty,\infty)$$



Example 7: Defining a Line Parametrically

b) Find the equation of the segment shown at the right.

Slope of Line?

$\frac{5}{4}$

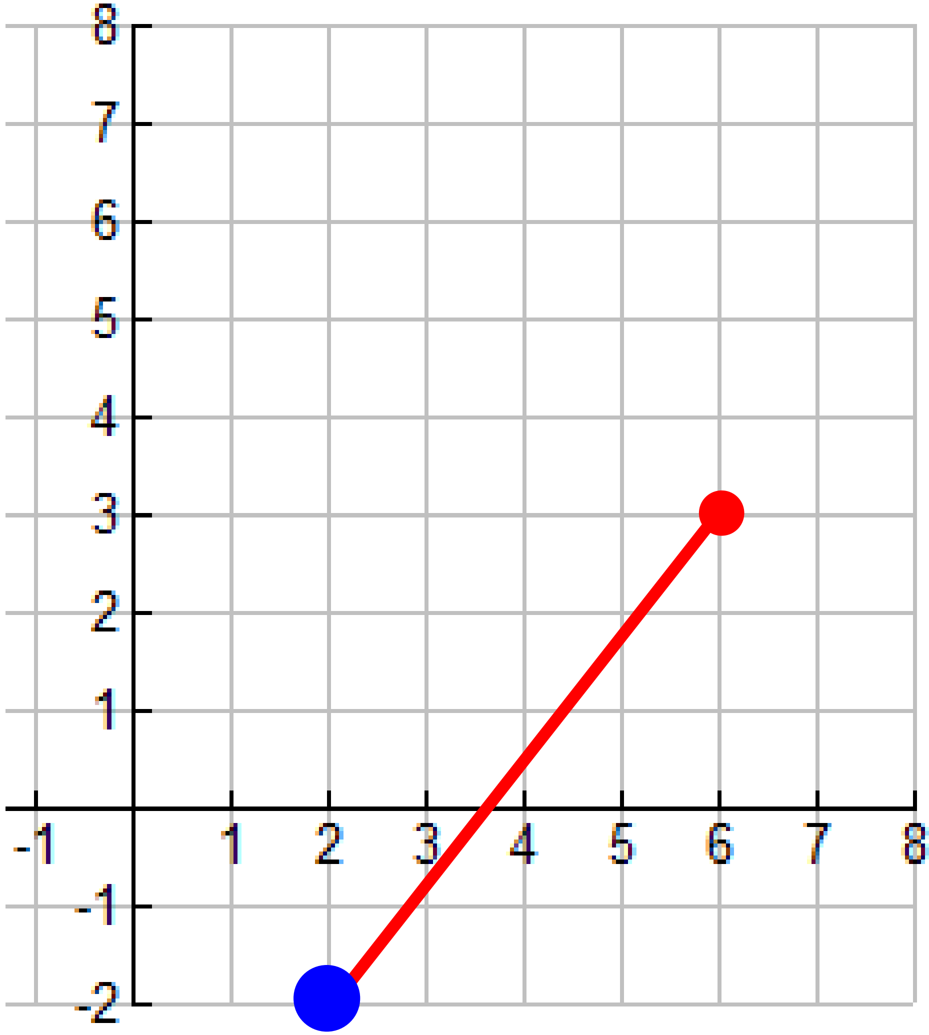
Vector Through Given Points?

$$(6,3)-(2,-2)=(4,5)$$

Equation of Line Segment:

$$L(t)=(2,-2)+t(4,5)$$

$$t\in[0,1]$$



Example 7: Defining a Line Parametrically

c) Describe a general formula :

Given two points on a line, P and Q, the parametric equation of the line between them is given by the following formula:

$$L(t) = P + (Q - P)t$$

Starting point (P)

Vector between the two points (Q − P)

Always specify the range for the parameter, whether it is $t \in (-\infty, \infty)$, $t \in [-1, 1]$, or something else.

Example 7: Defining a Line Parametrically

c) Describe a general formula :

Given two points on a line, P and Q, the parametric equation of the line between them is given by the following formula:

$$L(t) = P + (Q - P)t$$

Starting point (P)

Vector between the two points (Q − P)

To what formula from Algebra 1 is this one analogous? Why? Describe the corresponding parts.

Example 8: x-y-z Equations

Find the xyz-equations of the line through (2,1,0) and (6,0,6)

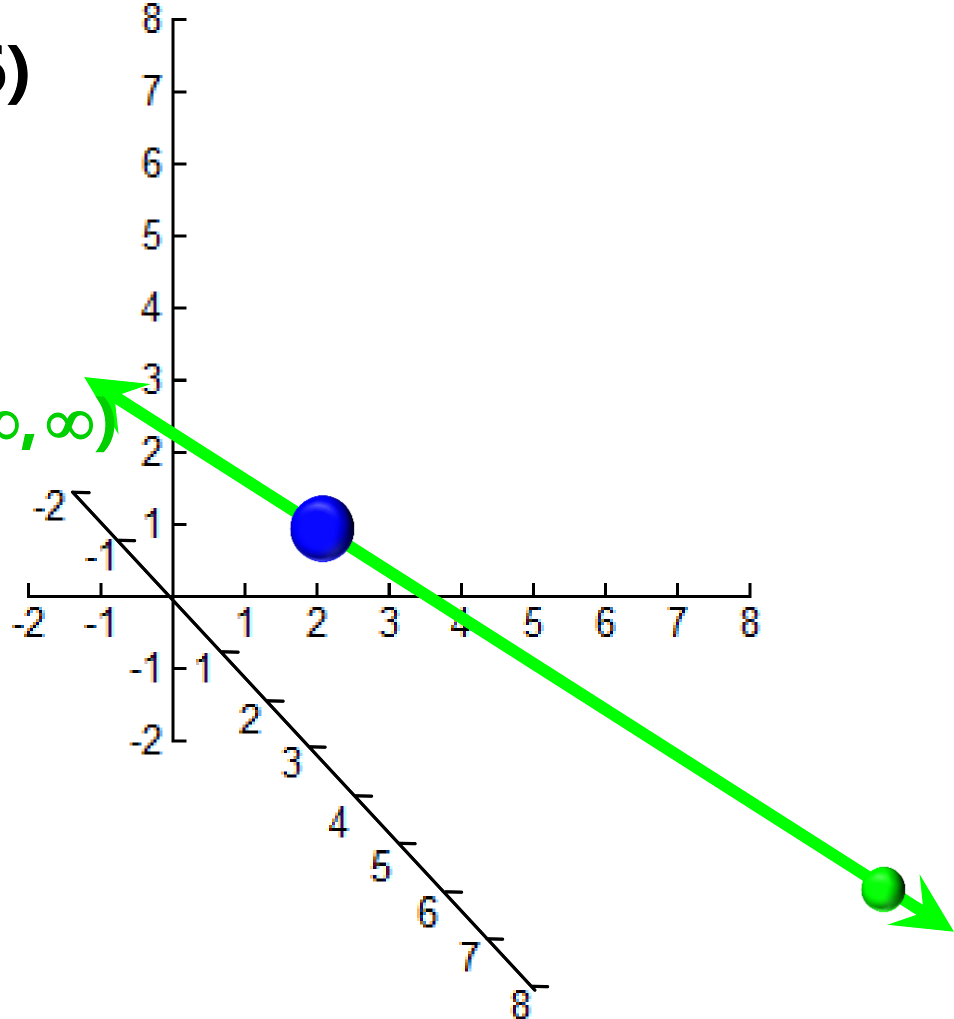
Vector:

$$(6,0,6)-(2,1,0)=(4,-1,6)$$

Parametric Equation :

$$L(t)=(2,1,0)+t(4,-1,6),\ t\in(-\infty,\infty)$$

$$\begin{cases} x(t)=2+4t \\ y(t)=1-t \\ z(t)=0+6t \end{cases}$$

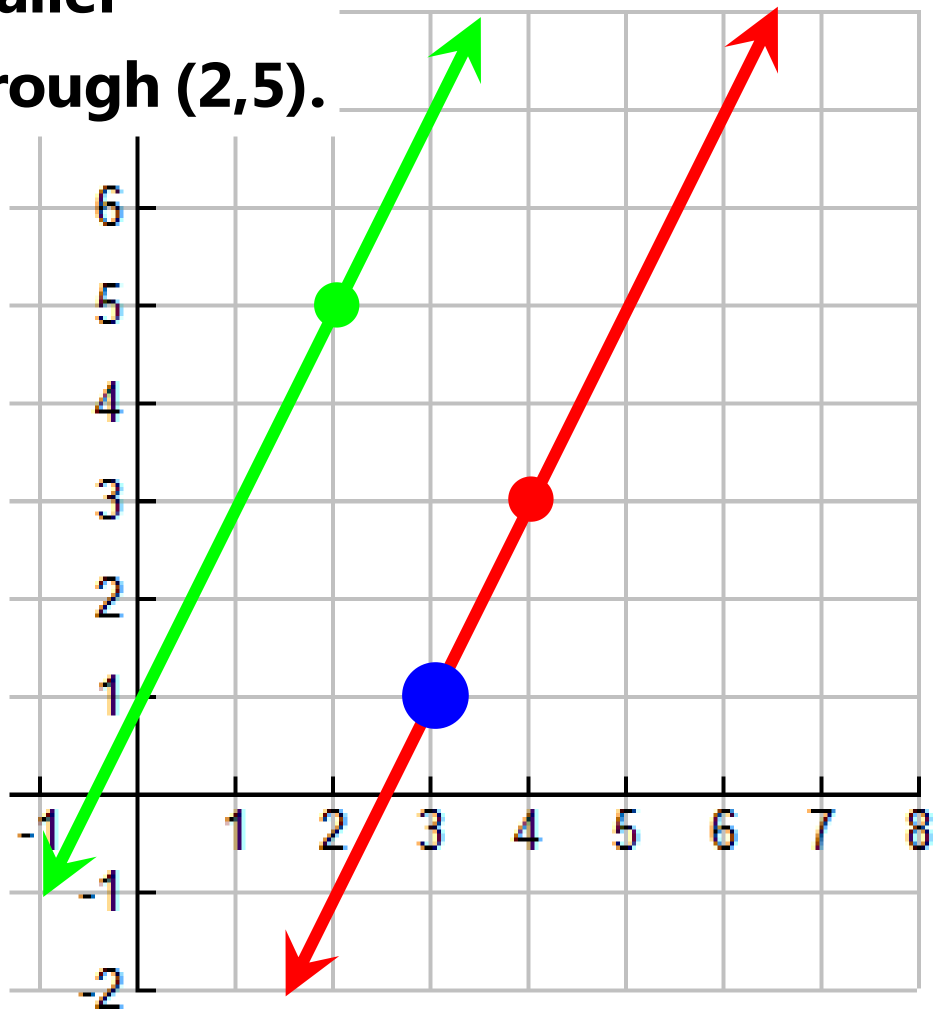


xyz-Equation:

$$\frac{x-2}{4}=1-y=\frac{z}{6}$$

Example 9: Defining a Line Parametrically

a) Find the equation of a line parallel to the one shown at the right through (2,5).



Slope of Line: $\frac{2}{1}$

Vector Through Given Points:

$$(4,3)-(3,1)=(1,2)$$

Equation of Line :

$$L(t)=(3,1)+t(1,2) \qquad t\in(-\infty,\infty)$$

Slope of Line: $\frac{2}{1}$

Vector Through Given Points:

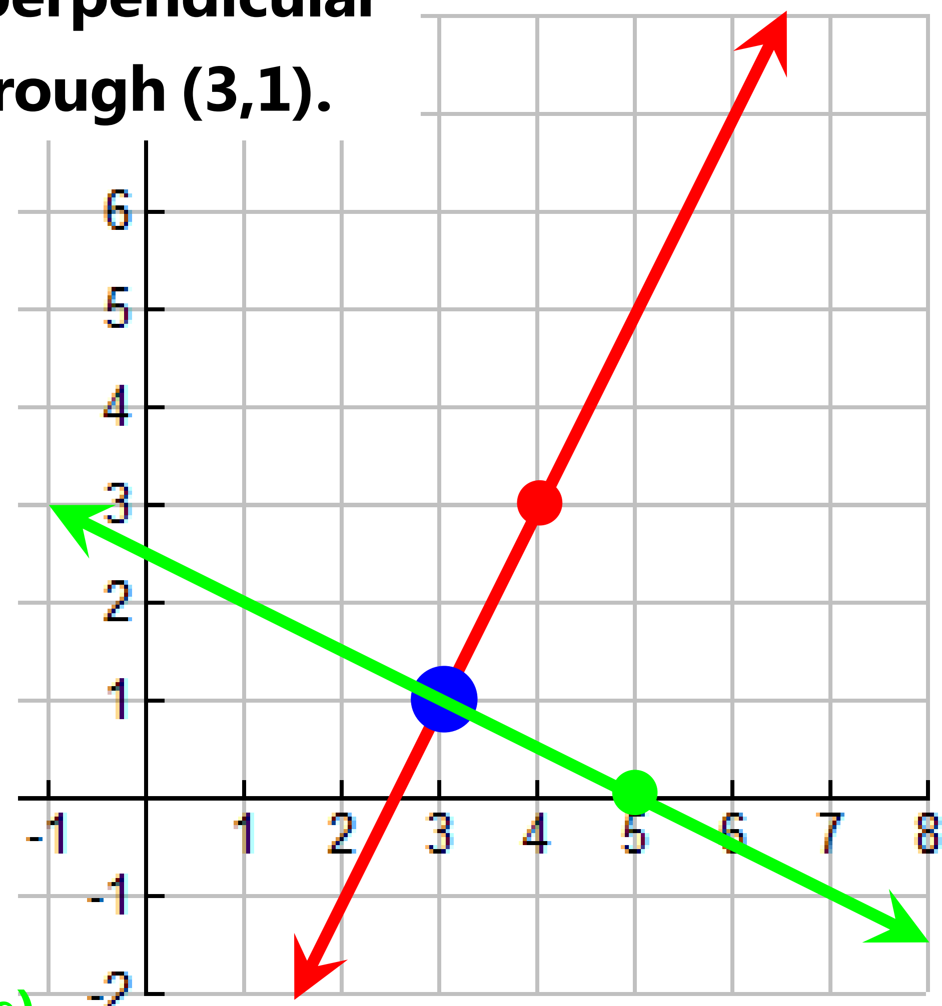
$$(1,2)$$

Equation of Line :

$$L(t)=(2,5)+t(1,2) \qquad t\in(-\infty,\infty)$$

Example 9: Defining a Line Parametrically

b) Find the equation of the line perpendicular to the one shown at the right through (3,1).



Slope of Line: $\frac{2}{1}$

Vector Through Given Points:

$$(4,3)-(3,1)=(1,2)$$

Equation of Line :

$$L(t)=(3,1)+t(1,2) \qquad t\in(-\infty,\infty)$$

Slope of Line: $-\frac{1}{2}$

Vector Through Given Points:

$$(5,0)-(3,1)=(2,-1)$$

Equation of Line :

$$L(t)=(3,1)+t(2,-1) \qquad t\in(-\infty,\infty)$$

Example 9: Defining a Line Parametrically

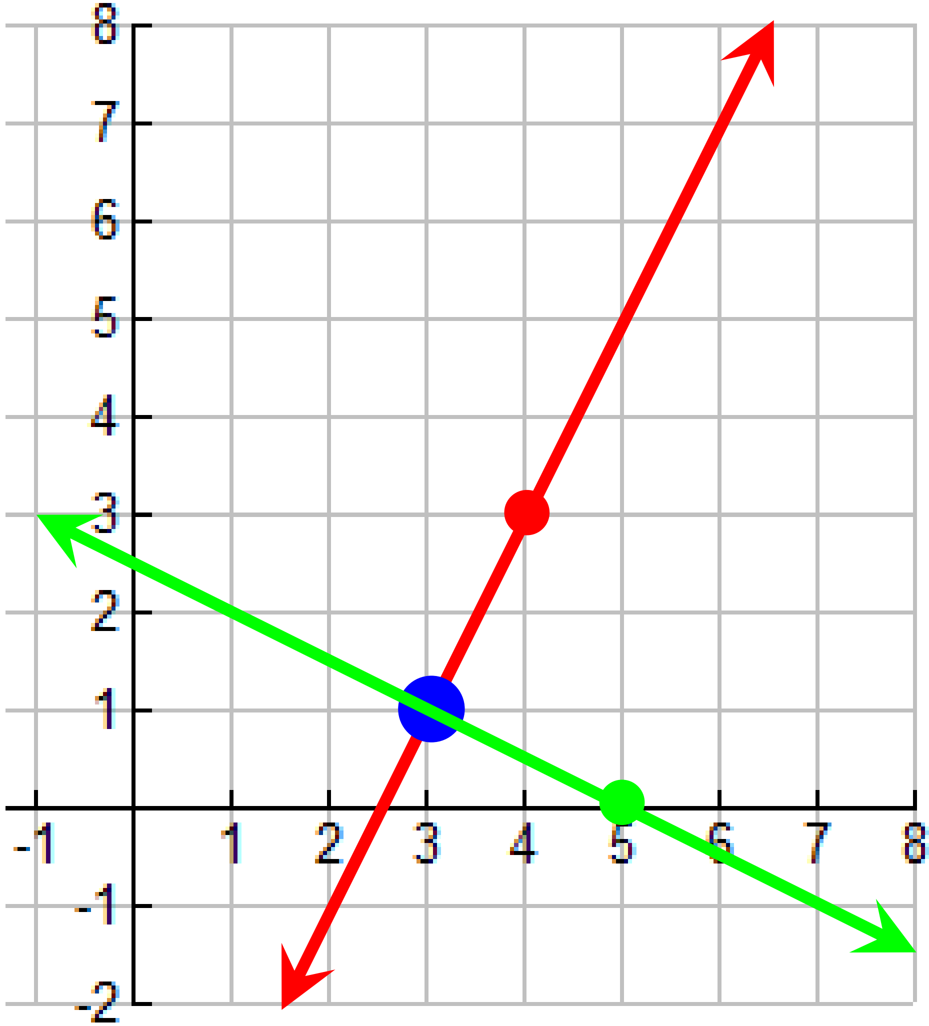
c) Rewrite the formula as $\begin{cases} x(t) \\ y(t) \end{cases}$

Equation of Line :

$$L(t) = (3,1) + t(2,-1)$$

$$t \in (-\infty, \infty)$$

$$\begin{cases} x(t) = 3 + 2t \\ y(t) = 1 - t \end{cases}$$



Example 9: Defining a Line Parametrically

Let a line through a point P be given by the following equation:

$$L(t) = P + t(a,b),\ t \in (-\infty, \infty)$$

The equation of the line parallel to L(t) through point R :

$$M(t) = R + t(a,b),\ t \in (-\infty, \infty)$$

$$\left(\text{such that } R \notin L(t)\right)$$

The equation of the line perpendicular to L(t) through point S :

$$N(t) = S + (b,-a)t,\ t \in (-\infty, \infty)$$

Defining a Line in 3D-space Parametrically

Let a line through a point $P = (x_0, y_0, z_0)$ be given by the following equation where $V = (v_1, v_2, v_3)$ is a generating (direction) vector:

$$L(t) = P + tV,\ t \in (-\infty, \infty)$$

The equation of the line parallel to $L(t)$ through point R :

$$M(t) = R + tV,\ t \in (-\infty, \infty)$$

$\left(\text{such that } R \notin L(t)\right)$

The equation of the line perpendicular to $L(t)$ through point S :

$$N(t) = S + tW,\ t \in (-\infty, \infty)$$

$\left(\text{such that } L(t) \text{ and } N(t) \text{ intersect and } V \bullet W = 0\right)$

Are the Following Pairs of Lines Perpendicular?

$L(t) = (1,4,2) + t(3,1,1)$
$M(t) = (1,4,2) + t(0,-1,1)$
$t \in (-\infty, \infty)$

L(t) and M(t) intersect at (1,4,2) and $(3,1,1) \bullet (0,-1,1) = 0$, so these lines are $\perp$

$L(t) = (2,0,-1) + t(3,1,1)$
$M(t) = (5,1,0) + t(0,-1,1)$
$t \in (-\infty, \infty)$

The lines intersect at $L(1) = M(0) = (5,1,0)$ and $(3,1,1) \bullet (0,-1,1) = 0$, so these lines are $\perp$

$L(t) = (5,1,0) + t(3,1,1)$
$M(t) = (-3,1,2) + t(0,-1,1)$
$t \in (-\infty, \infty)$

L(t) and M(t) do not intersect, so these lines are not $\perp$ (they are skew)

Are the Following Pairs of Lines Parallel?

$L(t) = (1,4,2) + t(3,1,1)$

$M(t) = (5,9,6) + t(3,1,1)$

$t \in (-\infty, \infty)$

$(1,4,2) \notin M(t)$ and the lines have the same generating vector, so they are $\parallel$

$L(t) = (1,4,2) + t(3,1,1)$

$M(t) = (-2,3,1) + t(3,1,1)$

$t \in (-\infty, \infty)$

$M(1) = (1,4,2) \in L(t)$ and the lines have the same generating vector, so they are the same line twice.

$L(t) = (1,4,2) + t(3,1,1)$

$M(t) = (10,7,5) + t(6,2,2)$

$t \in (-\infty, \infty)$

$L(3) = (10,7,5) \in M(t)$ and the generating vectors of the lines are multiples of each other, so they are the same line twice.

Example 10: Unit Vectors

Let $f(t) = (x(t), y(t)) = (t^2, 5t - t^2)$ for $t \geq 0$.

a) Tangent vector at $t = 1$

(velocity vector):

$f'(t) = (2t, 5 - 2t) \Rightarrow f'(1) = (2, 3)$

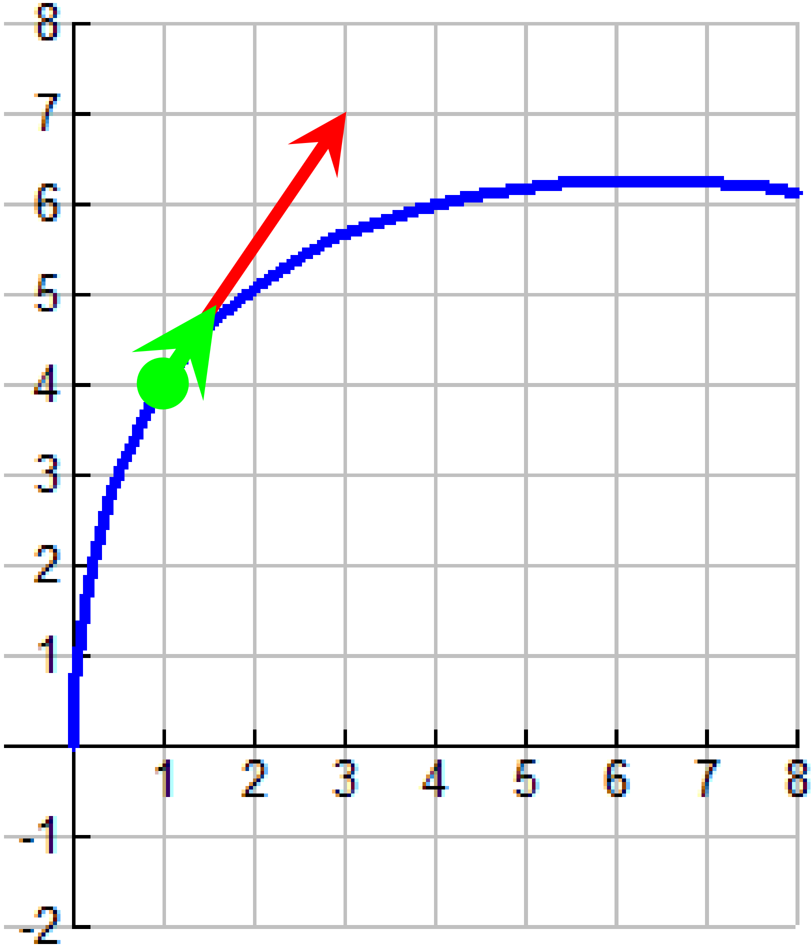
Tail is at $f(1) = (1, 4)$

b) Find the unit tangent vector at t=1:

Call $f'(1) = (2, 3)$ vector V.

$$\text{UnitTan} = \frac{V}{|V|}$$

$$\text{UnitTan} = \frac{(2,3)}{\sqrt{13}} \approx (0.555, 0.832)$$



c) How long is UnitTan? Why is it useful?

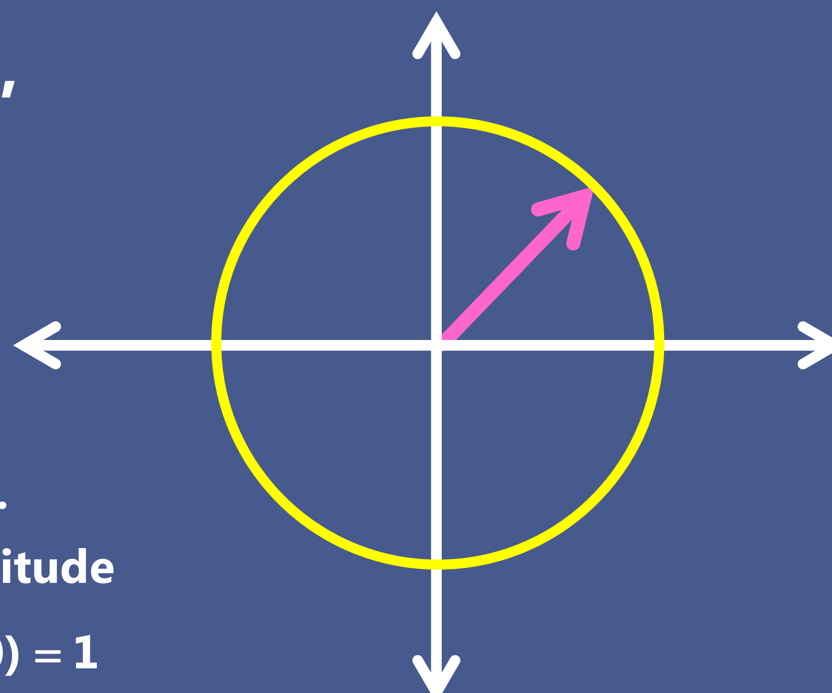
Example 10: Unit Vectors

Unit vector : A vector of length (magnitude) 1.

That is, if $\vec{u}$ is a unit vector, then $\left|\vec{u}\right| = 1$.

Fun fact: for any unit vector $\vec{u}$,
there exists some $\theta \in [0, 2\pi)$
such that $\vec{u} = (\cos(\theta), \sin(\theta))$.

Proof: First, you know that $(\cos(\theta), \sin(\theta))$
traces out every direction $\vec{u}$ could point in.
Now you know $(\cos(\theta), \sin(\theta))$ has a magnitude
of 1 since $\left|(\cos(\theta), \sin(\theta))\right| = \cos^2(\theta) + \sin^2(\theta) = 1$



Example 10: Unit Vectors

Given a vector V, you can find a unit vector in the direction of V as follows :

$$\mathbf{UnitVector} = \frac{\mathbf{V}}{|\mathbf{V}|}$$

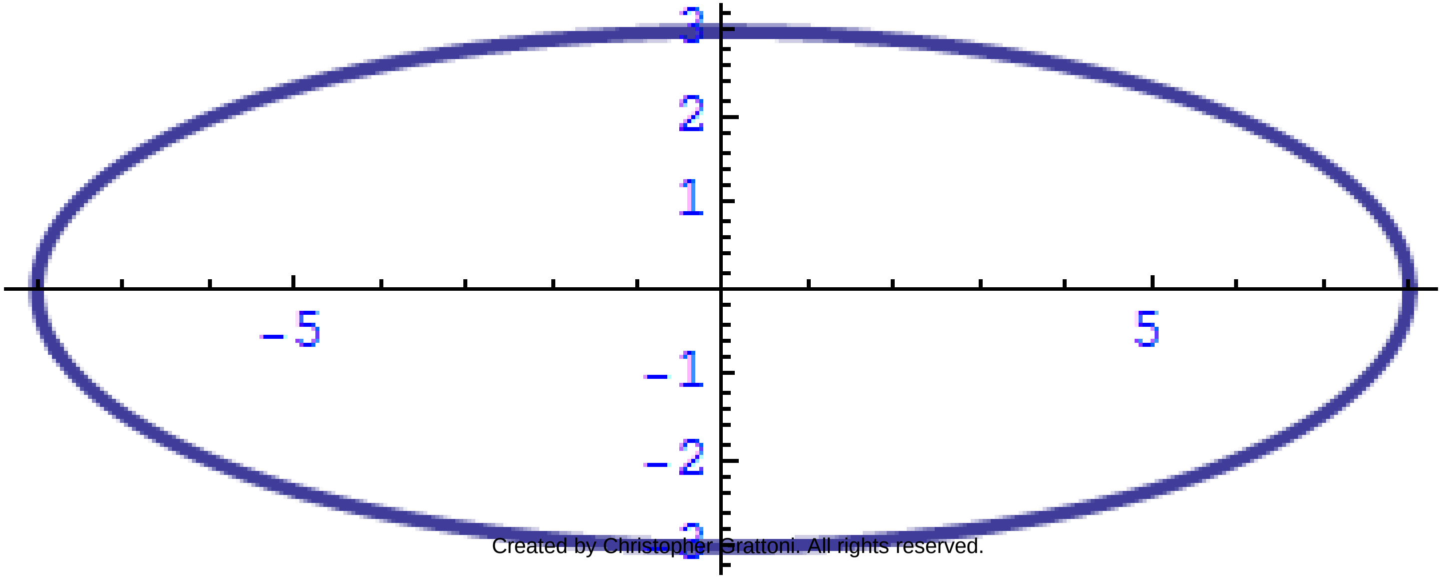
This is known as "normalizing" vector V

This encodes direction information without any of the magnitude distractions.

Example 11: Particle Path, Velocity, and Acceleration

A particle's position is described by the equation:

$$P(t) = (8\cos(t), 3\sin(t)) \quad 0 \le t < 2\pi$$



Example 11: Particle Path, Velocity, and Acceleration

A particle's position is described by the equation:

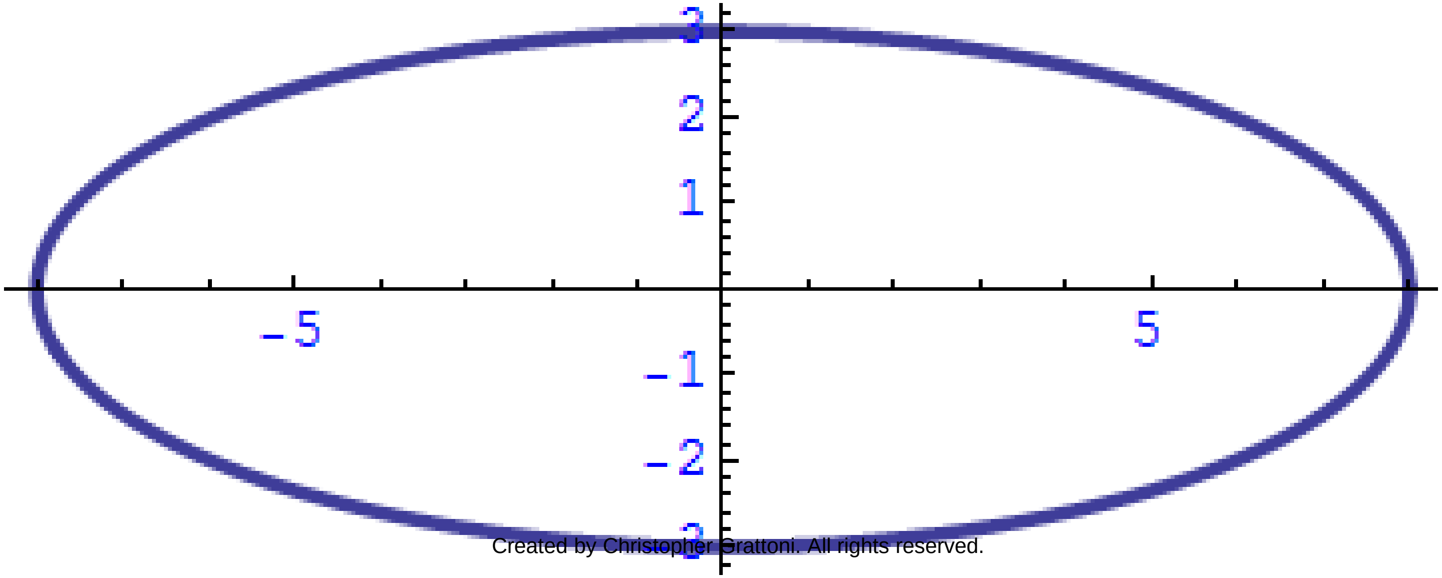
$$P(t) = (8\cos(t), 3\sin(t)) \quad 0 \le t < 2\pi$$

Its velocity is the derivative of the position function:

$$P'(t) = v(t) = (-8\sin(t), 3\cos(t))$$

Its acceleration is the derivative of the velocity function:

$$P''(t) = a(t) = (-8\cos(t), -3\sin(t))$$



Example 11: Particle Path, Velocity, and Acceleration

Velocity and acceleration vectors at $t = 0s$:

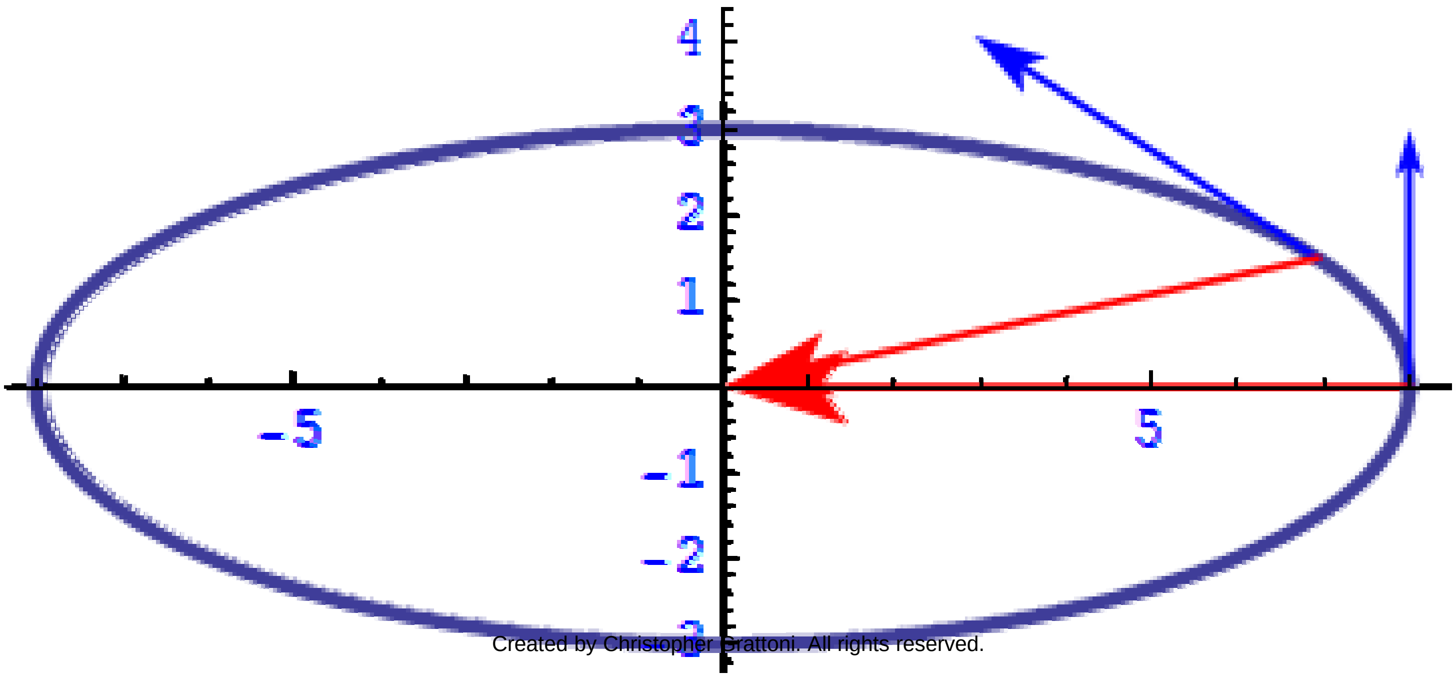
$$P(t) = (8\cos(t), 3\sin(t))$$

$$v(t) = (-8\sin(t), 3\cos(t))$$

$$a(t) = (-8\cos(t), -3\sin(t))$$

At $t = \frac{\pi}{6}s$:

At various other t values :



Example 11: Particle Path, Velocity, and Acceleration

Velocity and acceleration vectors at $t = 0s$:

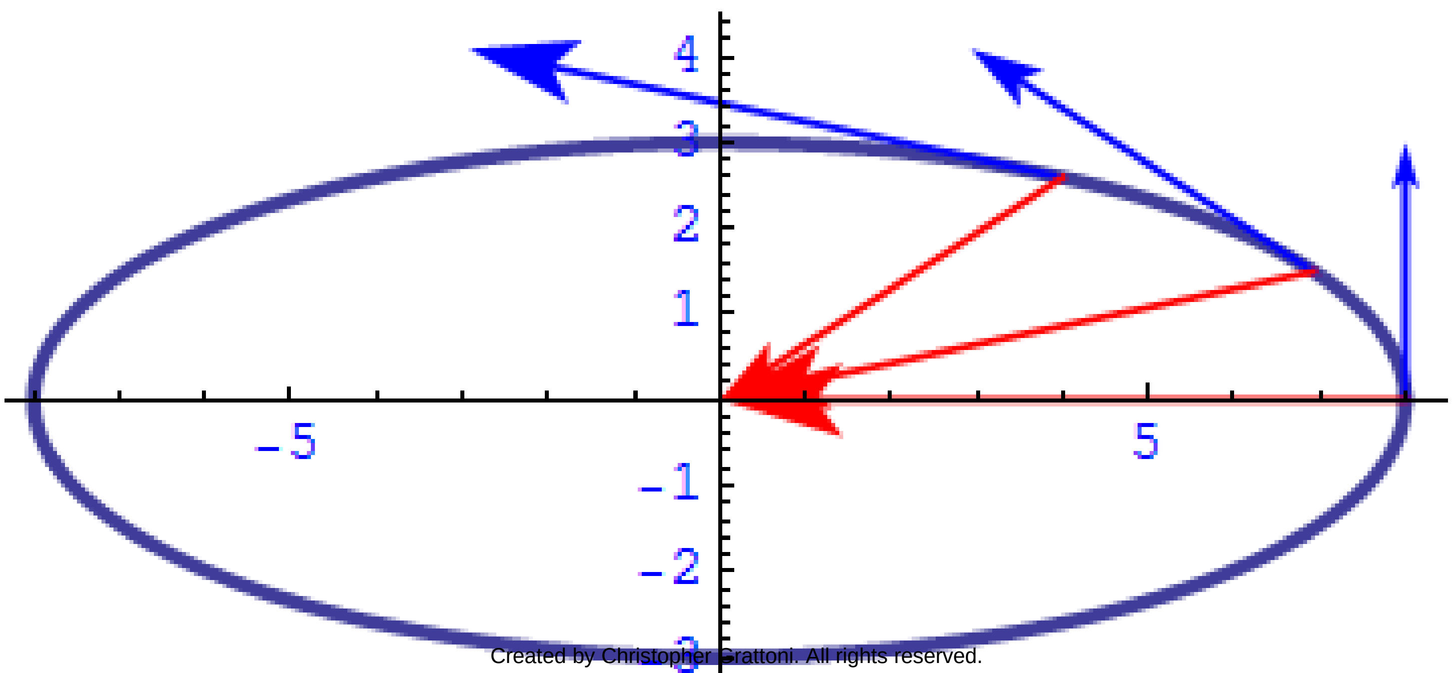
At $t = \frac{\pi}{6}s$:

At various other t values:

$$P(t) = (8\cos(t), 3\sin(t))$$

$$v(t) = (-8\sin(t), 3\cos(t))$$

$$a(t) = (-8\cos(t), -3\sin(t))$$



Example 1: Particle Path, Velocity, and Acceleration

Velocity and acceleration vectors at $t = 0s$:

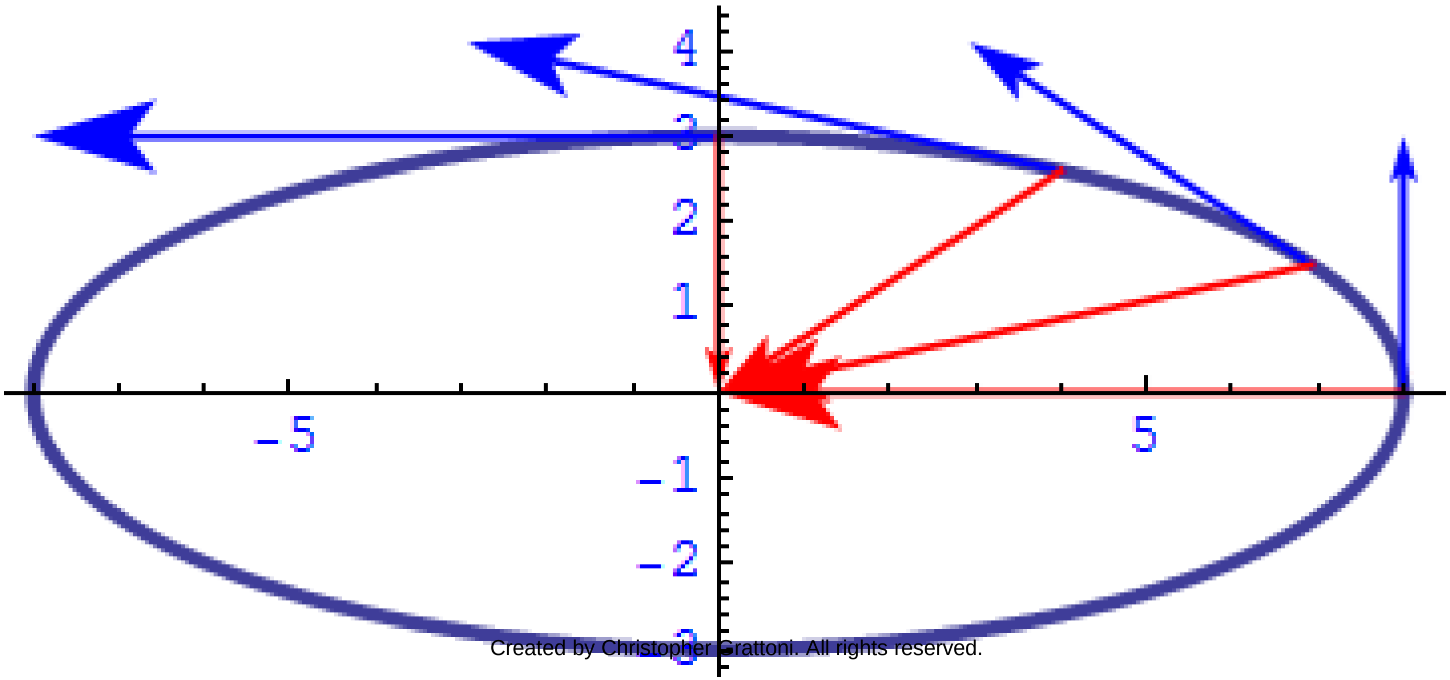
At $t = \frac{\pi}{6}s$:

At various other t values:

$$P(t) = (8\cos(t), 3\sin(t))$$

$$v(t) = (-8\sin(t), 3\cos(t))$$

$$a(t) = (-8\cos(t), -3\sin(t))$$



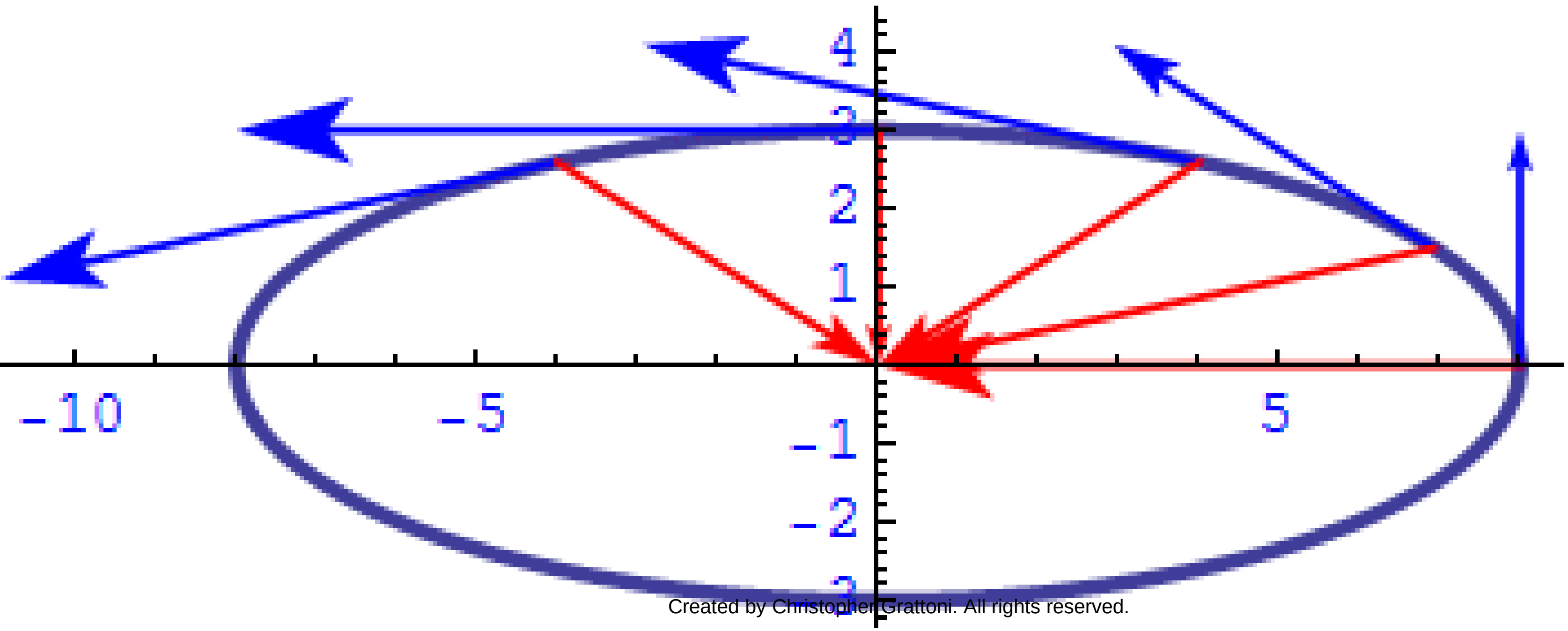
Example 1: Particle Path, Velocity, and Acceleration

$$P(t) = (8\cos(t), 3\sin(t))$$
$$v(t) = (-8\sin(t), 3\cos(t))$$
$$a(t) = (-8\cos(t), -3\sin(t))$$

Velocity and acceleration vectors at $t = 0s$:

At $t = \frac{\pi}{6} s$:

At various other t values:



Example 11: Particle Path, Velocity, and Acceleration

$$P(t) = (8\cos(t), 3\sin(t))$$

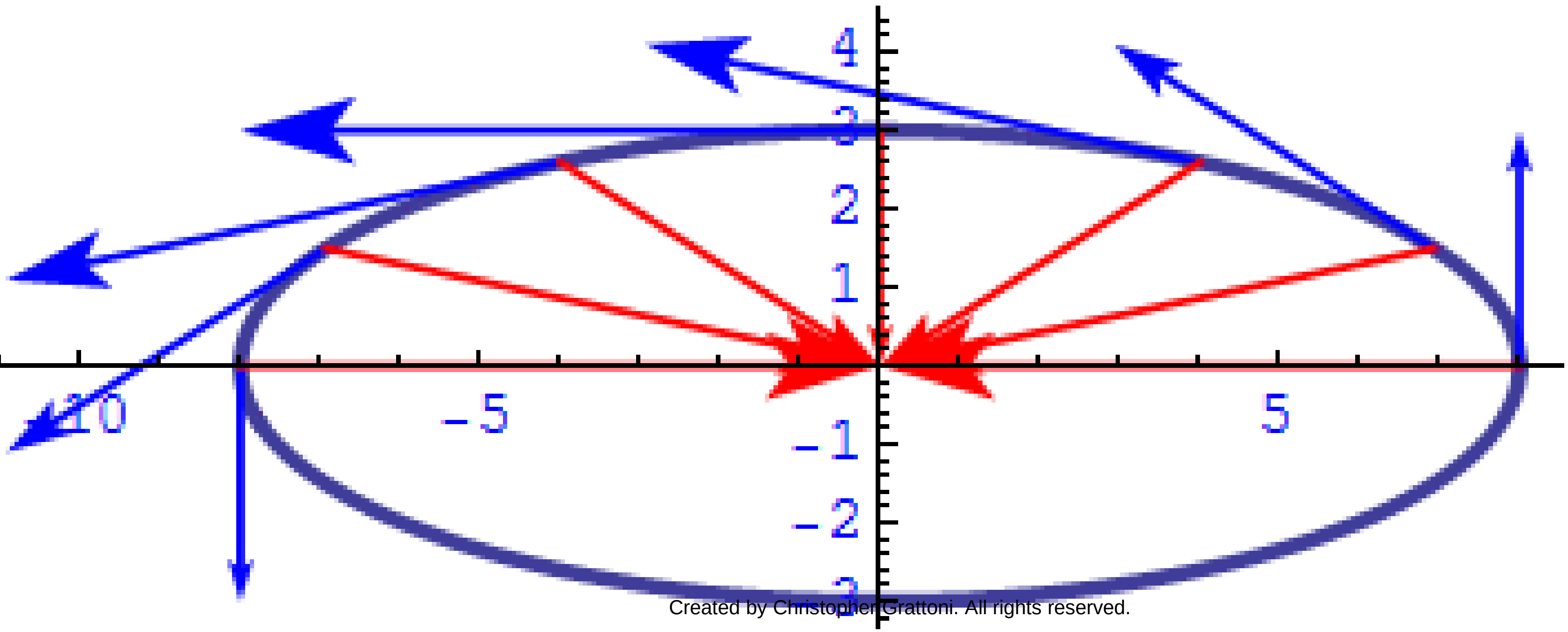
$$v(t) = (-8\sin(t), 3\cos(t))$$

$$a(t) = (-8\cos(t), -3\sin(t))$$

Velocity and acceleration vectors at $t = 0s$:

At $t = \frac{\pi}{6}s$:

At various other t values :



Example 11: Particle Path, Velocity, and Acceleration

$P(t) = (8\cos(t), 3\sin(t))$

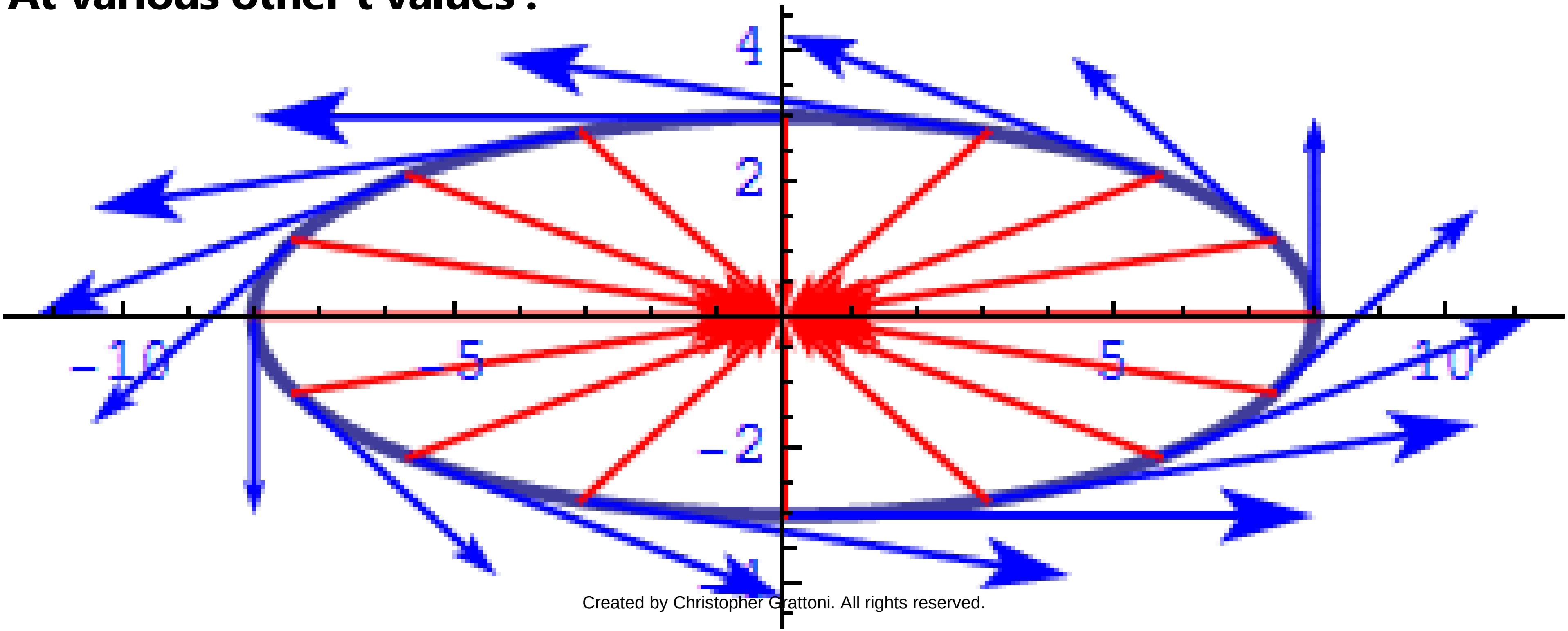
$v(t) = (-8\sin(t), 3\cos(t))$

$a(t) = (-8\cos(t), -3\sin(t))$

Velocity and acceleration vectors at $t = 0s$:

At $t = \frac{\pi}{6}s$:

At various other t values:



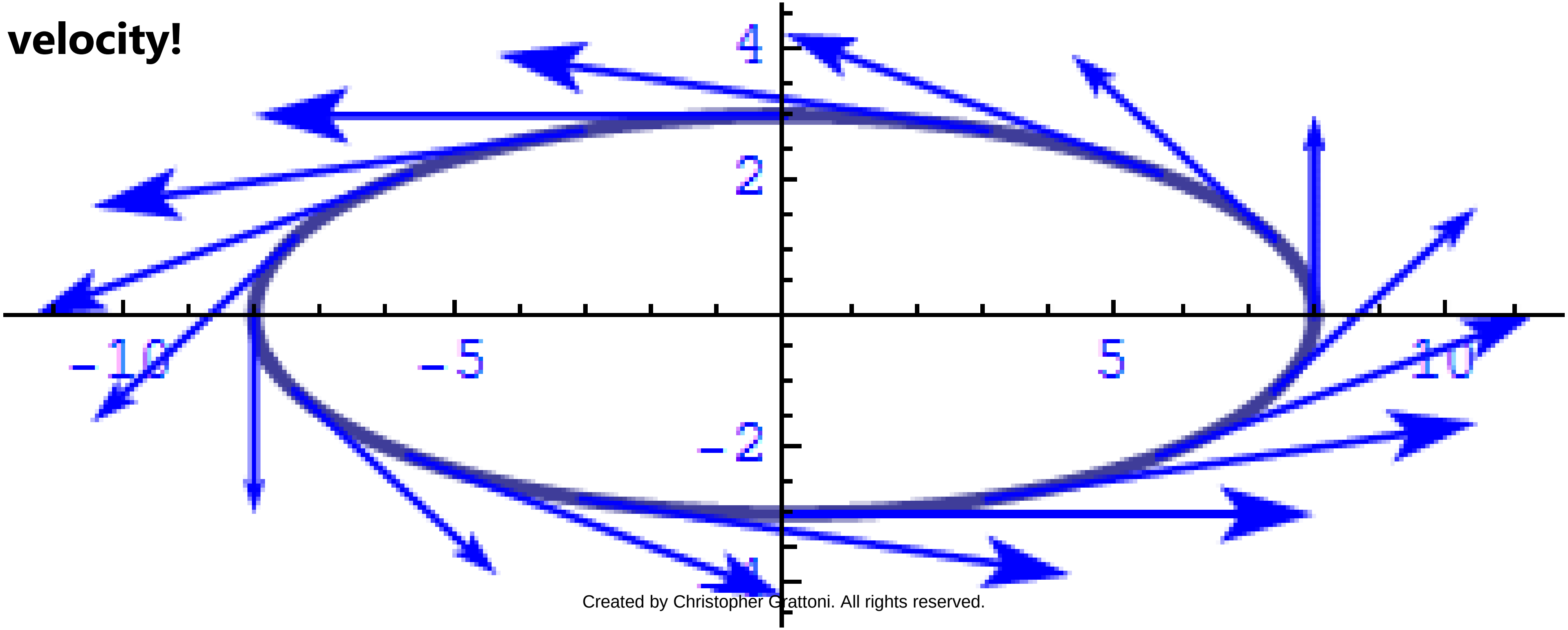
Example 11: Particle Path, Velocity, and Acceleration

Describe the velocity of the particle for $0 \le t < 2\pi$

How does the velocity vector aid this description?

$$P(t) = (8\cos(t), 3\sin(t))$$
$$v(t) = (-8\sin(t), 3\cos(t))$$
$$a(t) = (-8\cos(t), -3\sin(t))$$

Without getting fingerprints on the screen, trace the path of the particle paying close attention to accurately representing its velocity!



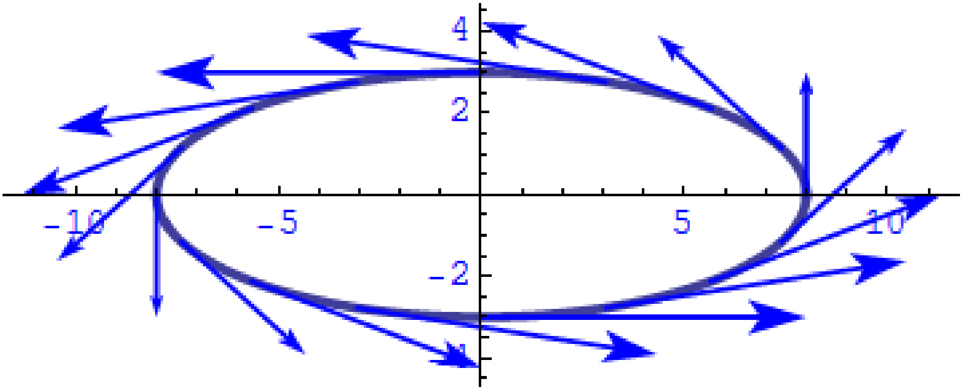
Example 11: Particle Path, Velocity, and Acceleration

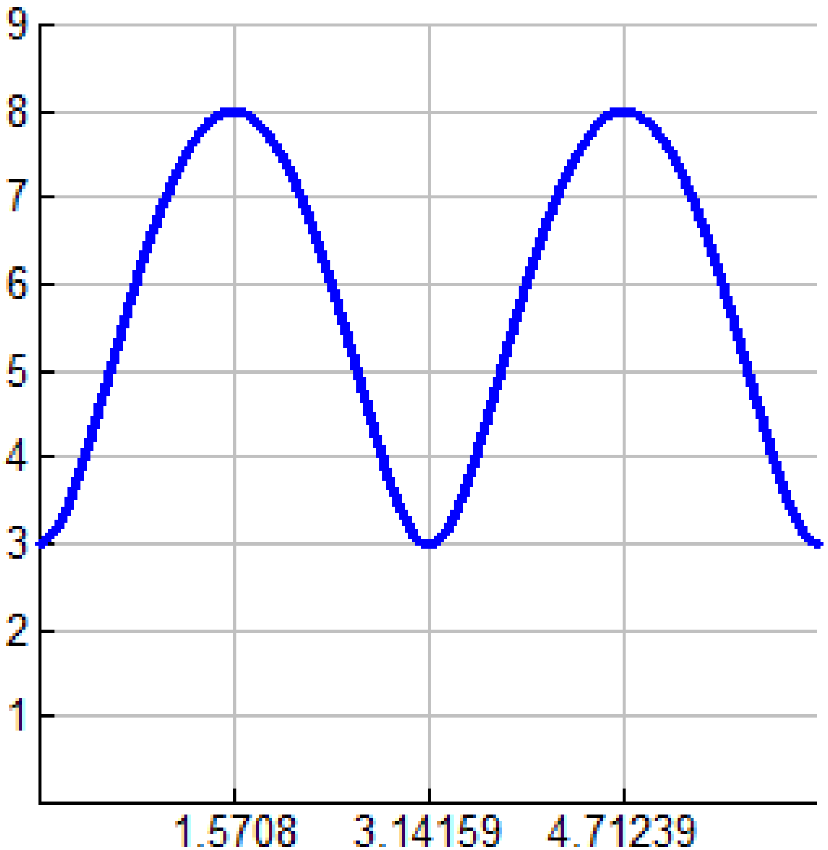
Speed is $s(t) = |v(t)|$, $0 \le t < 2\pi$, use this to help our description from previous slide :

$$P(t) = (8\cos(t), 3\sin(t))$$
$$v(t) = (-8\sin(t), 3\cos(t))$$
$$a(t) = (-8\cos(t), -3\sin(t))$$

$$s(t) = |v(t)| = \sqrt{v(t) \bullet v(t)} = \sqrt{64\sin^2(t) + 9\cos^2(t)}$$

How does speed appear in our plot from the previous slide?



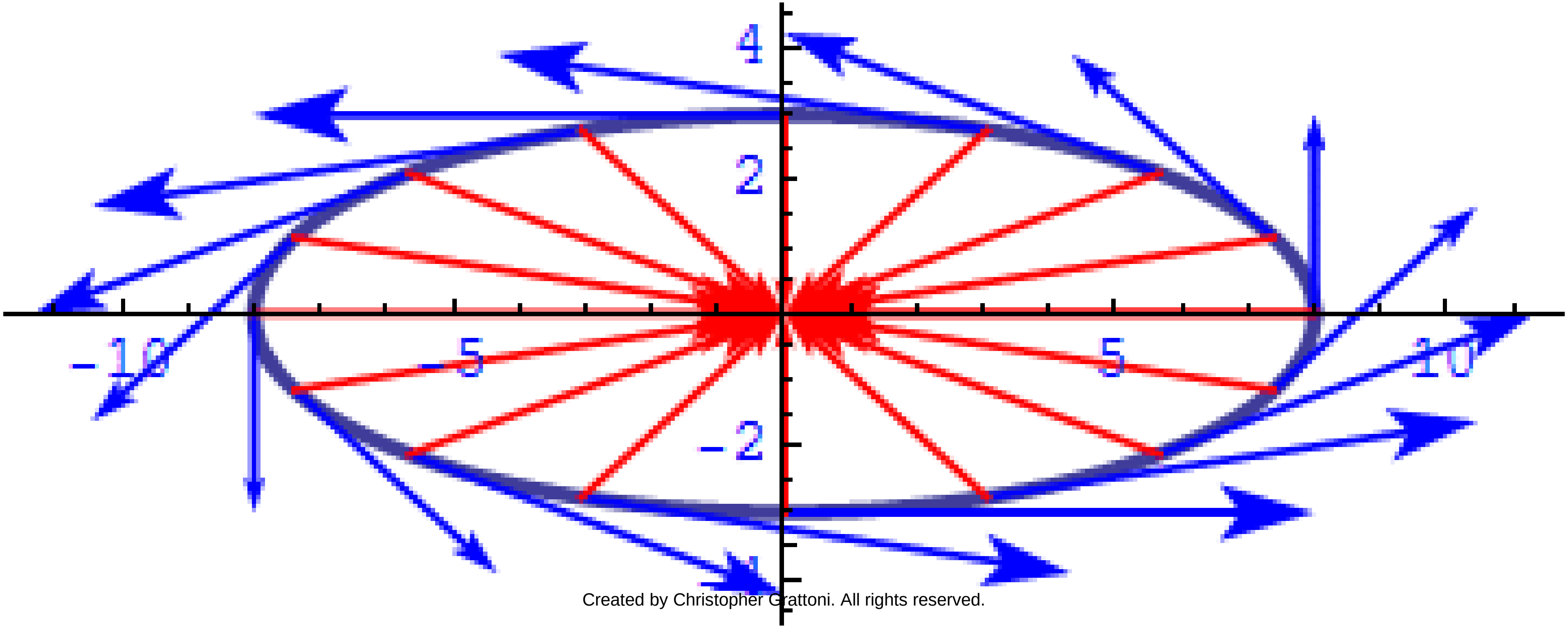


Example 11: Particle Path, Velocity, and Acceleration

Why do the acceleration vectors point inward?

If this particle were a train on an elliptical track, describe how you'd experience these acceleration vectors as a passenger on the train.

$$P(t) = (8\cos(t), 3\sin(t))$$
$$v(t) = (-8\sin(t), 3\cos(t))$$
$$a(t) = (-8\cos(t), -3\sin(t))$$



Example 11: Particle Path, Velocity, and Acceleration

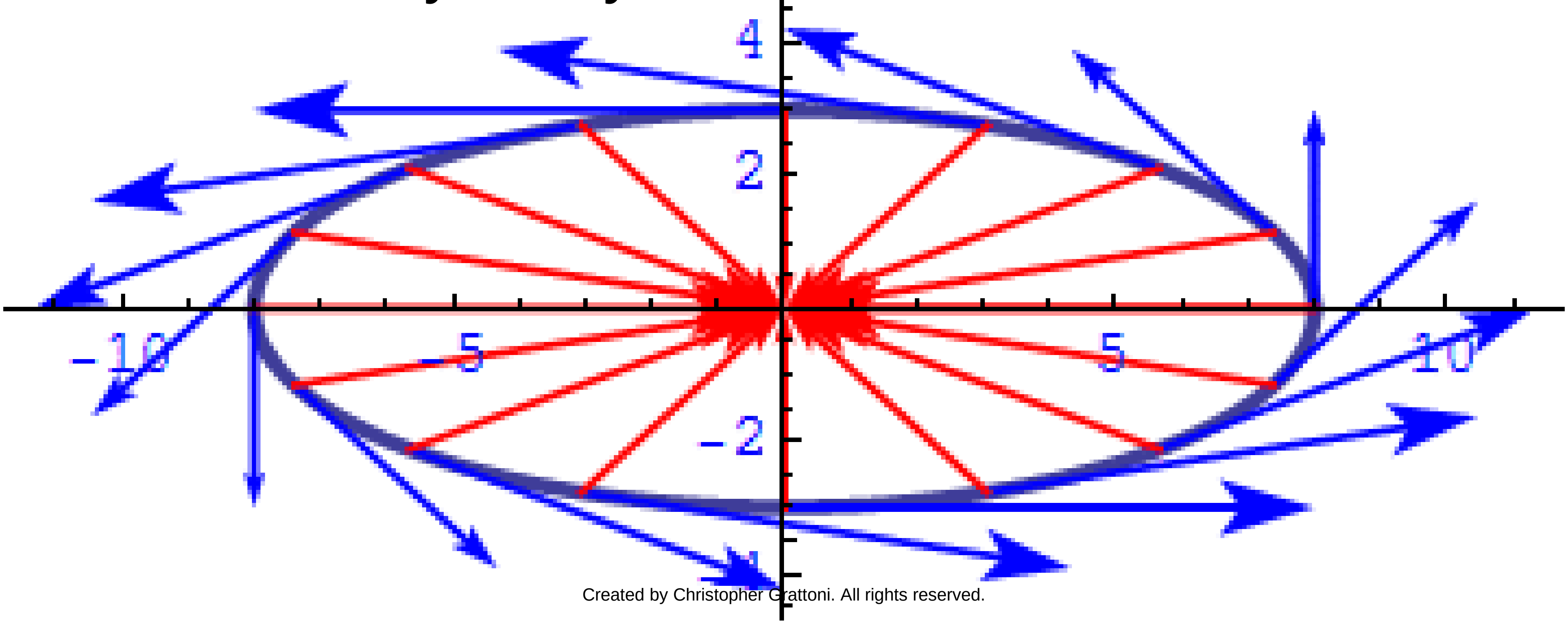
If the train suddenly derailed, would it continue around the ellipse? If not, in which direction would it go?

$P(t) = (8\cos(t), 3\sin(t))$

$v(t) = (-8\sin(t), 3\cos(t))$

$a(t) = (-8\cos(t), -3\sin(t))$

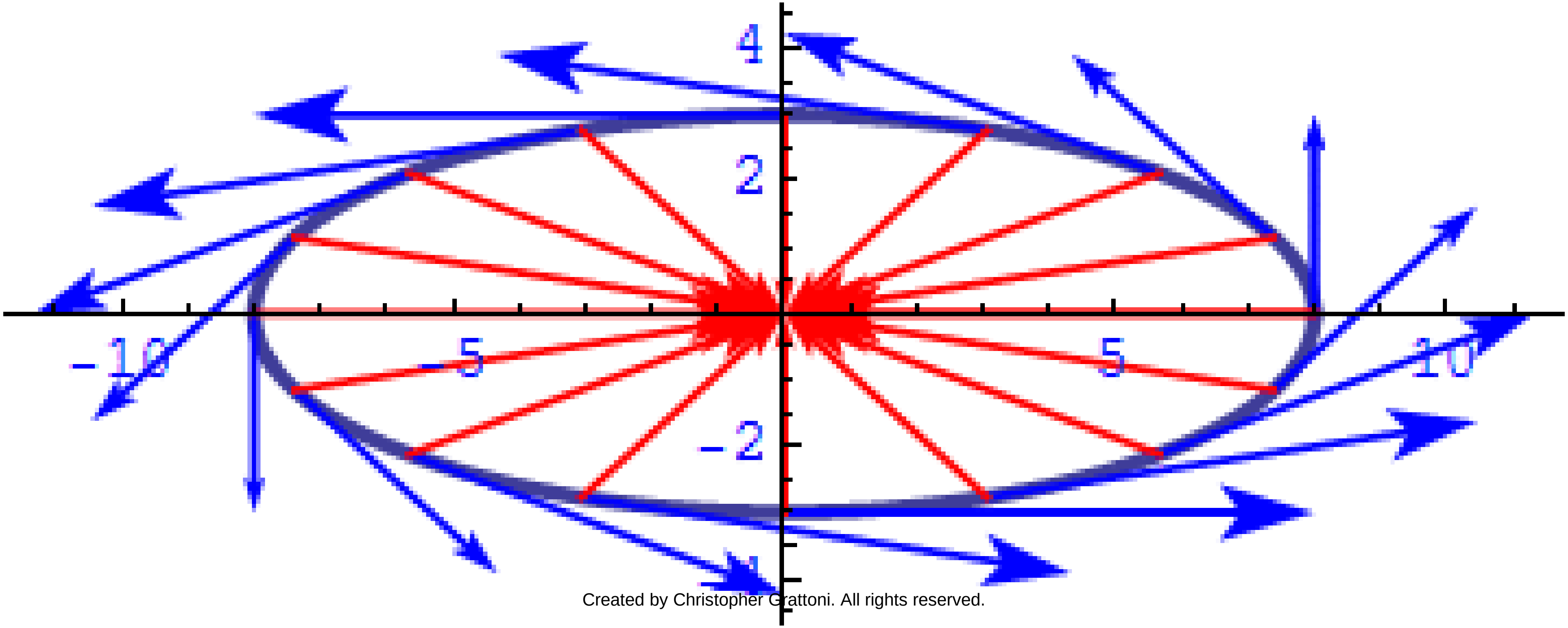
Are the velocity/acceleration vectors always perpendicular? When are/aren't they. Prove your answer.



Example 11: Particle Path, Velocity, and Acceleration

Are the velocity/acceleration vectors always ⊥ ?
When are/aren't they. Prove your answer.

$$P(t) = (8\cos(t), 3\sin(t))$$
$$v(t) = (-8\sin(t), 3\cos(t))$$
$$a(t) = (-8\cos(t), -3\sin(t))$$



Example 11: Particle Path, Velocity, and Acceleration

Read the textbook carefully today or tonight. It has more than I have included here, this is just a preview!