Lesson 3: Linearization Supplement

3D Plotting of Surfaces, Paths on Surfaces, and Linearization

Example 1: Linearizing y = f(x) at the Point (a,f(a))

Recall: The equation of a tangent line to a curve y = f(x) at the point (a, f(a)):

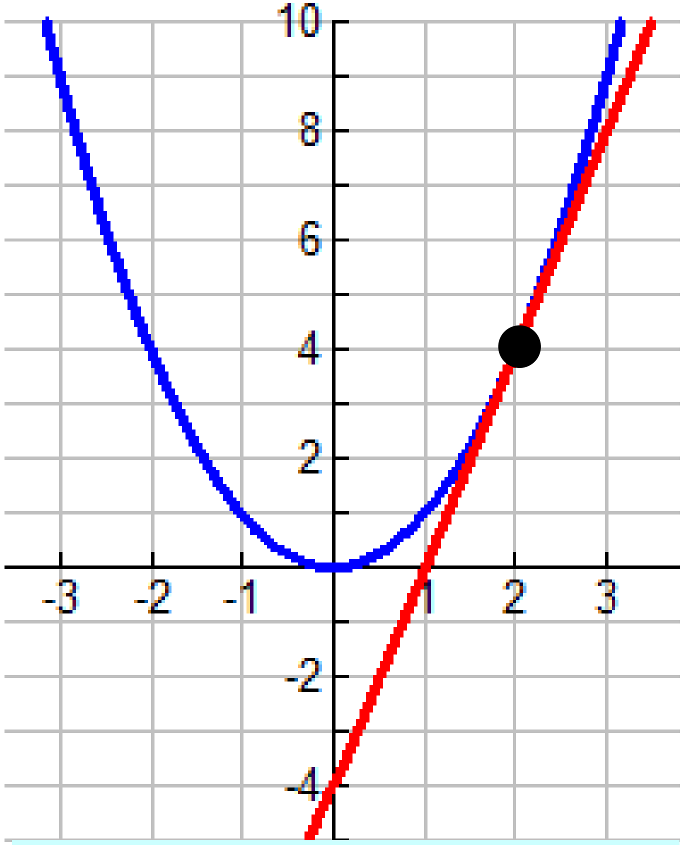
Point Slope Form: $y - f(a) = f'(a)(x - a)$

Linearization Form: $L(x) = f'(a)(x - a) + f(a)$

Remember that we can use this tangent line as a reasonable approximation of the curve near the point (a,f(a)).

See for yourself :

x	f(x)	L(x)	%Error
2	4	4	0
2.1	4.41	4.40	0.23%
2.5	6.25	6.00	4%
3	9	8	11.11%
6	36	20	44.44%



$f(x) = x^2$ at the point (2, f(2)):

$y - 4 = 4(x - 2)$

$L(x) = 4x - 4$

Example 2: Linearizing z = f(x,y) at the Point (a,b,f(a,b))

We would like to extend this idea to plotting a plane tangent to a 3D surface at a point, but we have a technical problem to overcome:

$$\text{First (Wrong) Try}: \ z - f(a,b) = f'(a,b)(x-a) + f'(a,b)(y-b)$$

This is a good start: We can see that this is an equation for a plane that passes through the point (a,b,f(a,b)). But $f'(a,b)$ should worry you...

What does $f'(a,b)$ mean if f(x,y) has two variables, x and y?

Really, we want to build our plane in two directions: x and y.
We need machinery to accomodate this need:

$$z - f(a,b) = \underbrace{f'(a,b)(x-a)}_{\text{x-direction}} + \underbrace{f'(a,b)(y-b)}_{\text{y-direction}}$$

Detour: The Partial Derivative

This is an easier fix than you'd think. We need the partial derivative of f(x,y) with respect to x or y, depending on the situation:

Partial derivative of f(x, y) with respect to x:

$$\frac{\partial f}{\partial x} = f_x(x,y) = f^{(1,0)}(x,y) = D[f[x,y],x]$$

(Hold y constant and take the derivative with respect to x.)

Partial derivative of f(x, y) with respect to y:

$$\frac{\partial f}{\partial y} = f_y(x,y) = f^{(0,1)}(x,y) = D[f[x,y],y]$$

(Take the derivative with respect to y and hold x constant.)

Try this for $f(x, y) = y^2 \cos(x)$:

$$\frac{\partial f}{\partial x} = -y^2 \sin(x)$$

$$\frac{\partial f}{\partial y} = 2y \cos(x)$$

Detour: Partial Versus Total Derivatives

Note that the partial differential, $\frac{\partial}{\partial x}$, behaves differently than the total differential, $\frac{d}{dx}$. Let's try this for $f(x, y) = x^2y^5 + 4y$:

Total Derivative (Treat y as a variable):

$$\frac{d}{dx}\left(x^2y^5 + 4y\right) = 2xy^5 + 5x^2y^4\frac{dy}{dx} + 4\frac{dy}{dx}$$

Partial Derivative (Hold y constant):

$$\frac{\partial}{\partial x}\left(x^2y^5 + 4y\right) = 2xy^5$$

Detour: Partial Versus Total Derivatives

Now try for $\frac{\partial}{\partial y}$ versus $\frac{d}{dy}$:

Total Derivative (Treat x as a variable):

$$\frac{d}{dy}\left(x^2y^5 + 4y\right) = 2xy^5\frac{dx}{dy} + 5x^2y^4 + 4$$

Partial Derivative (Hold x constant):

$$\frac{\partial}{\partial y}\left(x^2y^5 + 4y\right) = 5x^2y^4 + 4$$

Example 2: Linearizing z = f(x,y) at the Point (a,b,f(a,b))

With the partial derivative at our fingertips, we can now find the equation of a plane tangent to a surface:

Second Try : $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x(a, b)$ is slope of the tangent line in the x-direction for the surface $z = f(x, y)$ at $f(a, b)$

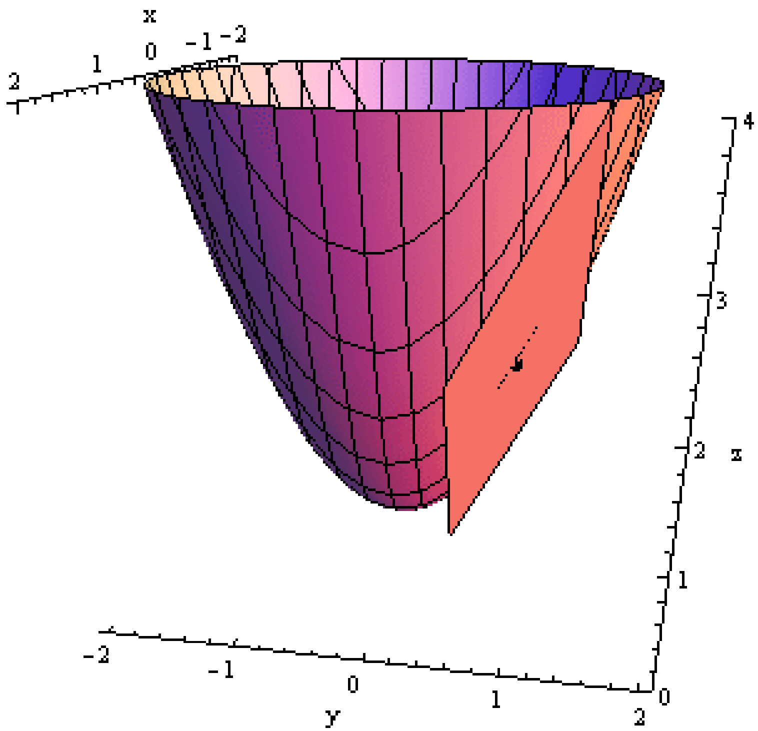
$f_y(a, b)$ is slope of the tangent line in the y-direction for the surface $z = f(x, y)$ at $f(a, b)$

Example 2: Linearizing z = f(x,y) at the Point (a,b,f(a,b))

Surface: $f(x,y) = x^2 + y^2$

Point: $(a,b) = (1,1)$

Find the equation of the tangent plane to f(x,y) at (1,1).



$$f_x(x,y) = 2x \Rightarrow f_x(1,1) = 2$$

$$f_y(x,y) = 2y \Rightarrow f_y(1,1) = 2$$

$$f(1,1) = 1^2 + 1^2 = 2$$

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

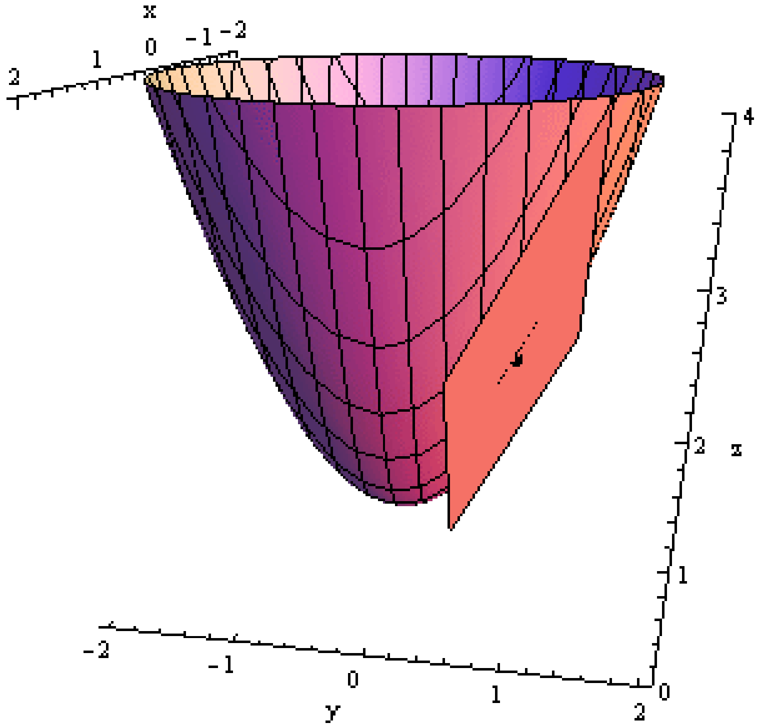
$$z - 2 = 2(x-1) + 2(y-1)$$

Example 2: Linearizing z = f(x,y) at the Point (a,b,f(a,b))

Surface: $f(x,y) = x^2 + y^2$

Point: $(a,b) = (1,1)$

Find a linearization, L(x,y), of f(x,y) at (1,1).



$$f_x(x,y) = 2x \Rightarrow f_x(1,1) = 2$$

$$f_y(x,y) = 2y \Rightarrow f_y(1,1) = 2$$

$$f(1,1) = 1^2 + 1^2 = 2$$

$$L(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$L(x,y) - 2 = 2(x-1) + 2(y-1)$$

Example 3: Using Linearization to Approximate a Path on a Surface

Putting it all together!

Surface:

$f(x,y) = x^2 + y^2$

Parametric Path on xy-plane:

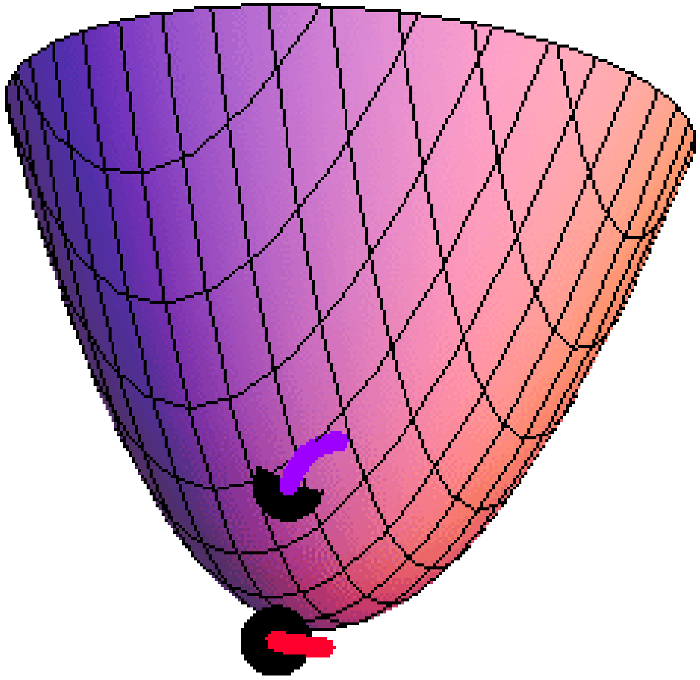
$(x(t), y(t)) = (0.7 + 4\sin(t), 0.4e^t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

$-0.1 \le t \le 0.1$



Find a linearization $L(x,y)$ of $f(x,y)$ at $(0.7, 0.4)$.

Use $L(x(t), y(t))$ to approximate $f(x(t), y(t))$.

Example 3: Using Linearization to Approximate a Path on a Surface

Putting it all together!

Surface:

$f(x, y) = x^2 + y^2$

Parametric Path on xy-plane:

$(x(t), y(t)) = (0.7 + 4\sin(t), 0.4e^t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

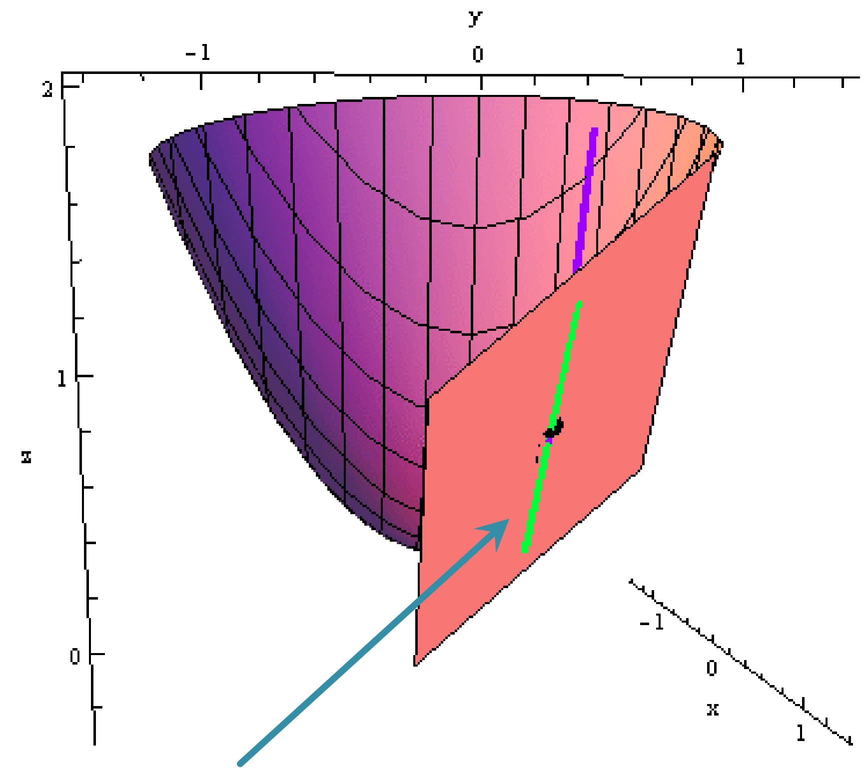
$-0.1 \leq t \leq 0.1$

Point of Linearization:

$(0.7, 0.4)$

Linearization:

$L(x, y) = f(0.7, 0.4) + f_x(0.7, 0.4)(x - 0.7) + f_y(0.7, 0.4)(y - 0.4)$



Path on Plane (Linearization):

$(x(t), y(t), L(x(t), y(t)))$

Example 4: Using Linearization to Approximate a Path on a Surface

Putting it all together!

Surface:

$f(x,y) = x^2 + y^2$

Parametric Path on xy-plane:

$(x(t), y(t)) = (0.7 + 4\sin(t), 0.4e^t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

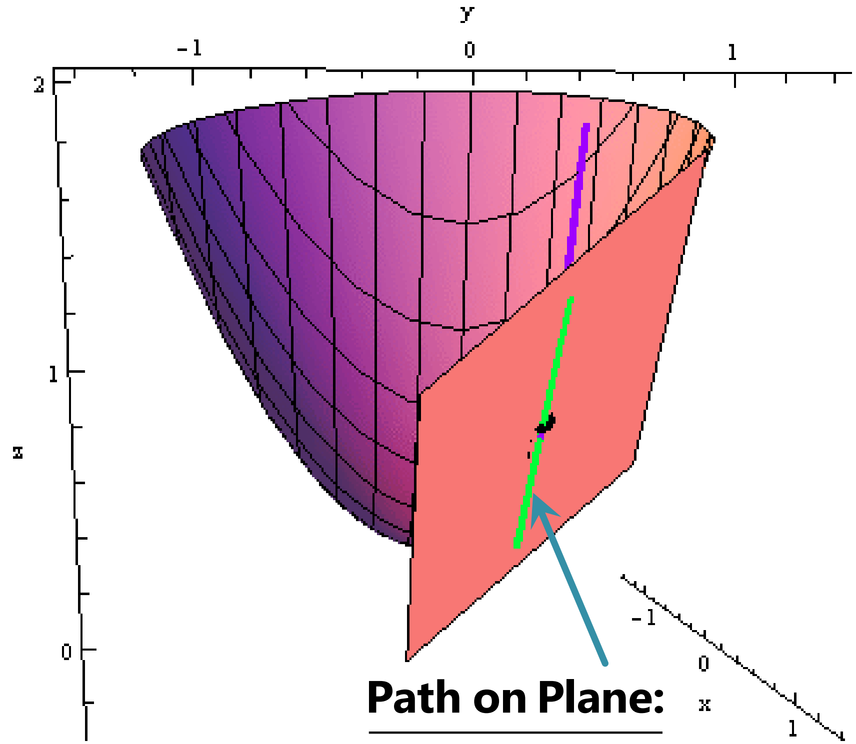
$-0.1 \le t \le 0.1$

Point of Linearization:

$(0.7, 0.4)$

Linearization:

$L(x,y) = f(0.7,0.4) + f_x(0.7,0.4)(x - 0.7) + f_y(0.7,0.4)(y - 0.4)$



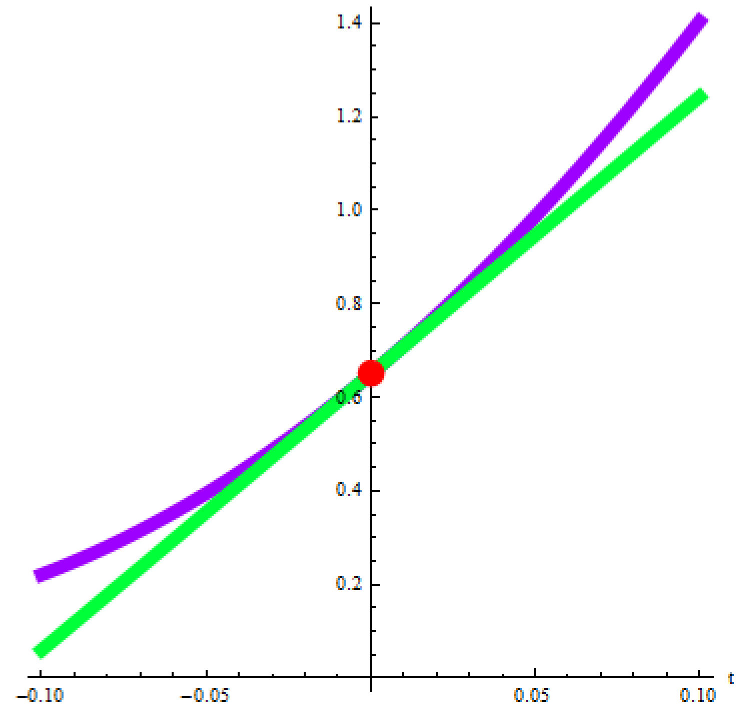
Path on Plane:

$(x(t), y(t), L(x(t), y(t)))$

How well does $L(x(t), y(t))$ approximate $f(x(t), y(t))$?

Example 4: Using Linearization to Approximate a Path on a Surface

We could get a handle on this by stripping out all the distractions in our previous graph. Instead, just put L(x(t),y(t)) versus t on a plot and f(x(t),y(t)) versus t on the same plot. Basically, just z-values versus time!



As expected, L(x(t), y(t)) = f(x(t), y(t) when t = 0 (at (0.7, 0.4)). It veers off from there, just like when we approximate y = f(x) near (a, f(a)) using the tangent to the curve at (a,f(a)).

Example 5: Using Linearization to Approximate Another Path on our Surface

Surface:

$f(x, y) = x^2 + y^2$

Parametric Path on xy-plane:

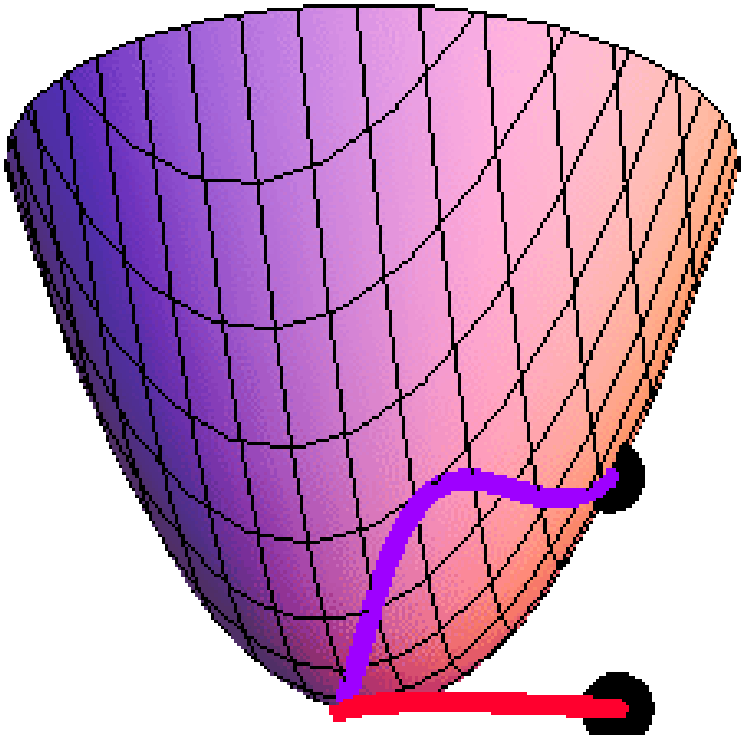
$(x(t), y(t)) = (0.7\cos(3t), 0.4 - t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

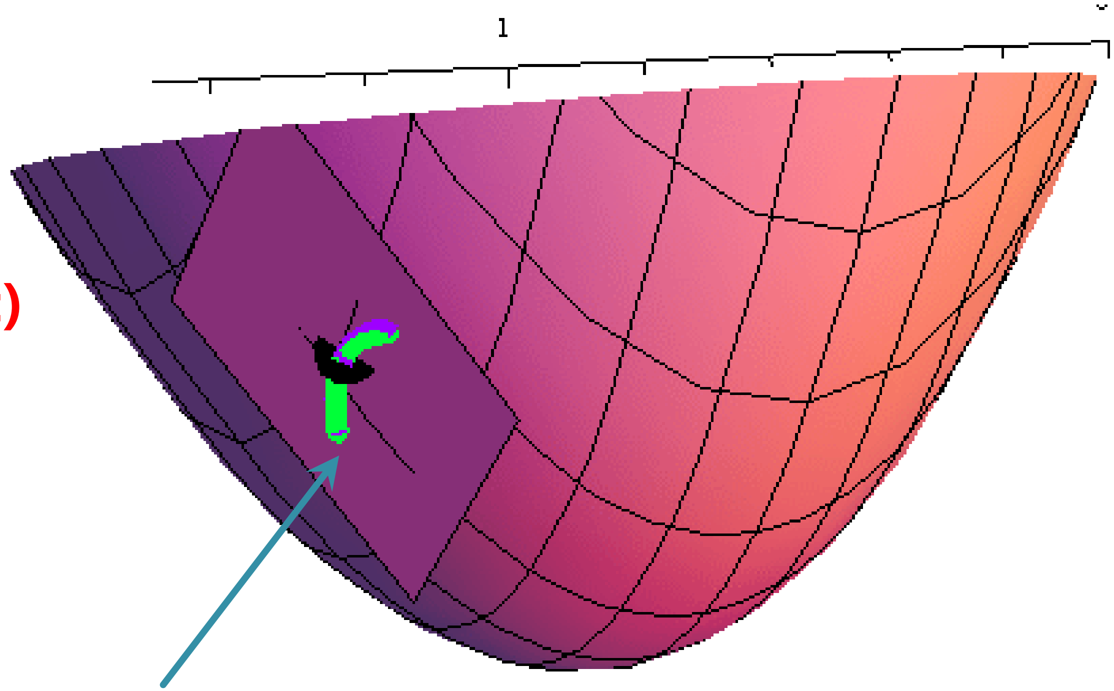
$-0.1 \le t \le 0.1$



Find a linearization $L(x, y)$ of $f(x, y)$ at $(0.7, 0.4)$.

Use $L(x(t), y(t))$ to approximate $f(x(t), y(t))$.

Example 5: Using Linearization to Approximate Another Path on our Surface



Surface:

$f(x, y) = x^2 + y^2$

Parametric Path on xy-plane:

$(x(t), y(t)) = (0.7\cos(3t), 0.4 - t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

$-0.1 \leq t \leq 0.1$

Point of Linearization:

$(0.7, 0.4)$

Path on Plane (Linearization):

$(x(t), y(t), L(x(t), y(t)))$

Linearization:

$L(x, y) = f(0.7, 0.4) + f_x(0.7, 0.4)(x - 0.7) + f_y(0.7, 0.4)(y - 0.4)$

Example 5: Using Linearization to Approximate Another Path on our Surface

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$f(x,y) = x^2 + y^2$

Parametric Path on xy-plane:

$(x(t), y(t)) = (0.7\cos(3t), 0.4 - t)$

Path on Surface:

$(x(t), y(t), f(x(t), y(t)))$

Time:

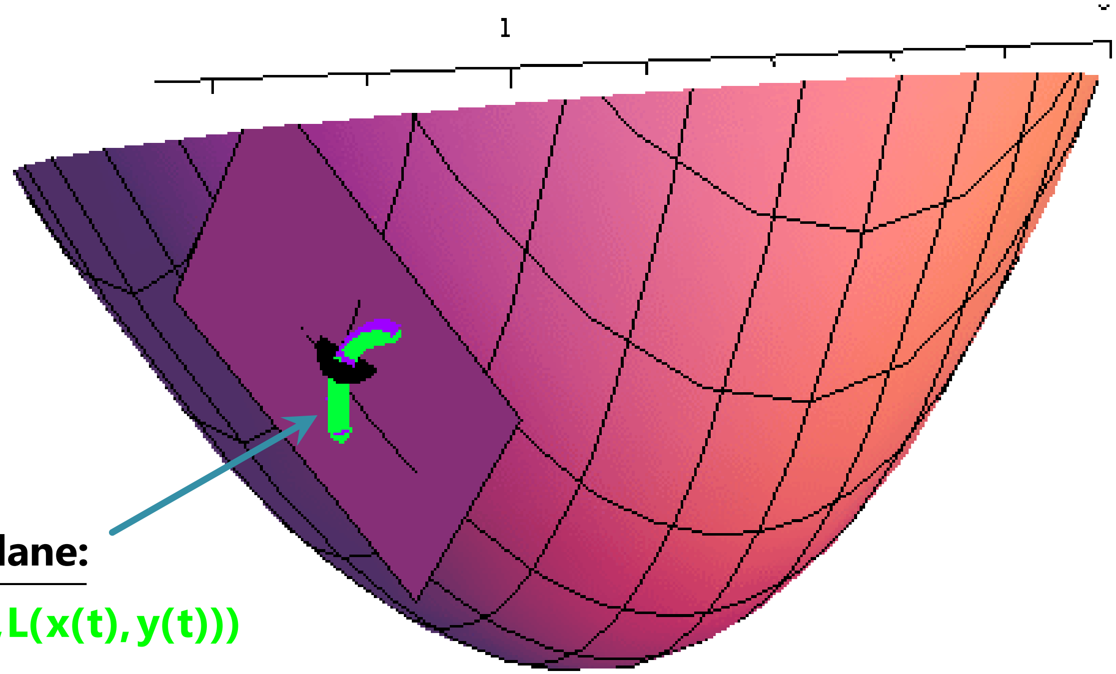
$-0.1 \le t \le 0.1$

Point of Linearization:

$(0.7, 0.4)$

Path on Plane:

$(x(t), y(t), L(x(t), y(t)))$



Linearization:

$L(x,y) = f(0.7,0.4) + f_x(0.7,0.4)(x - 0.7) + f_y(0.7,0.4)(y - 0.4)$

Notice that L(x,y) is the same as Example 7, but $(x(t), y(t), L(x(t), y(t)))$ is a different path on the plane from what we found before.

Example 5: Using Linearization to Approximate Another Path on our Surface

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$f(x,y) = x^2 + y^2$

Parametric Path on xy-plane:

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Point of Linearization:

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Path on Plane:

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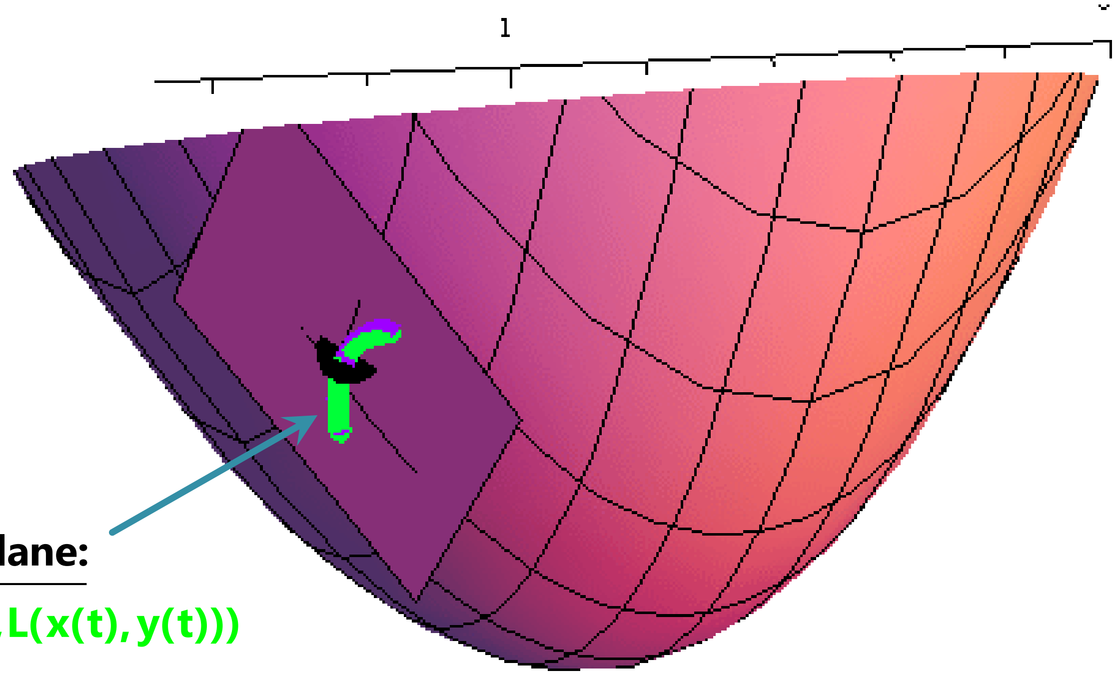
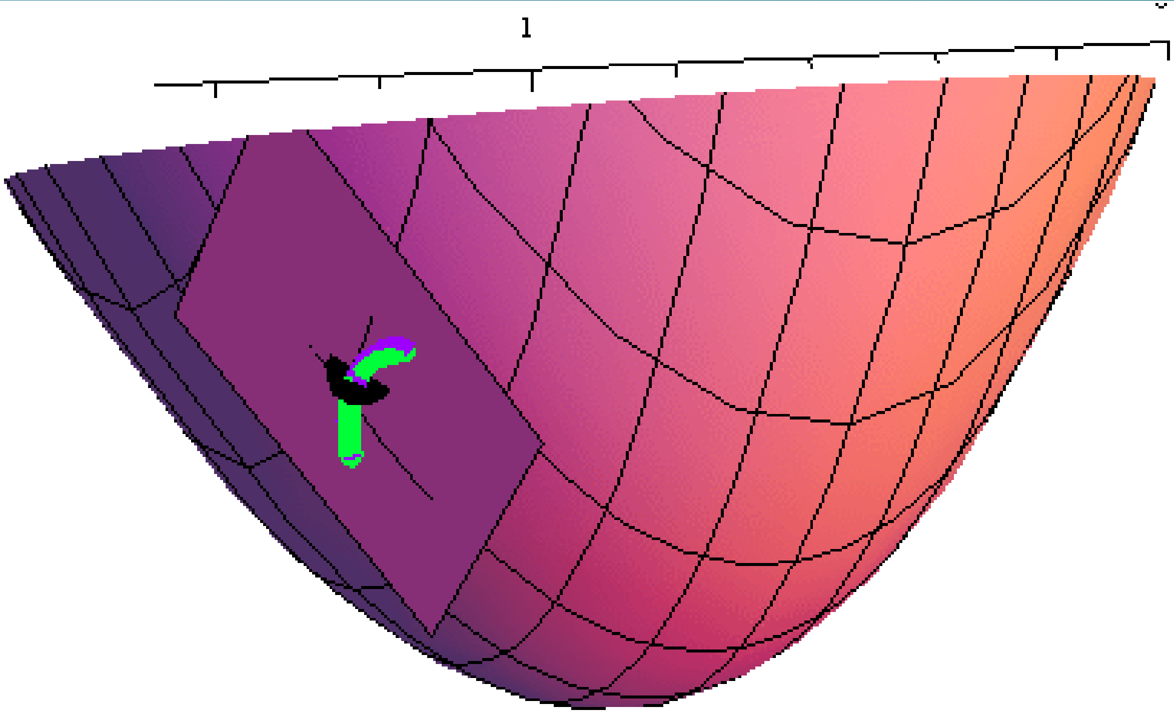


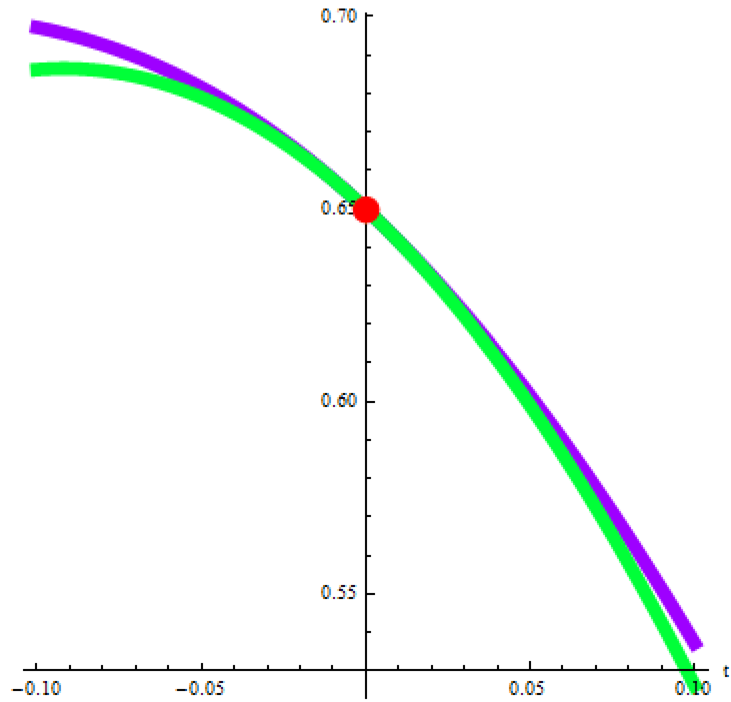
Linearization:

$L(x,y) = f(0.7,0.4) + f_x(0.7,0.4)(x-0.7) + f_y(0.7,0.4)(y-0.4)$

How well does $L(x(t), y(t))$ approximate $f(x(t), y(t))$?

Example 5: Using Linearization to Approximate Another Path on our Surface





You might be surpised to see L(x(t), y(t)) versus time is not linear.

You will be less surprised when you think of how (x(t), y(t), L(x(t), y(t))) is a non-linear path on a plane so your z-values versus time don't define a linear function.