Lesson 3

Perpendicularity, Planes, and Cross Products

Example 1: Equation for a Plane

Let P = (2,3,−1) be a point in space and let V = (4,−2,5) be a vector. Find the xyz-equation of the plane containing P that is perpendicular to vector V (that is, V is a normal vector to the plane).

The main idea here is that the vector V is perpendicular to the plane at ANY point on the plane.

Therefore, any vector with its tail at (2,3,-1) and tip at (x,y,z) on the plane will be perpendicular to V. This vector is:

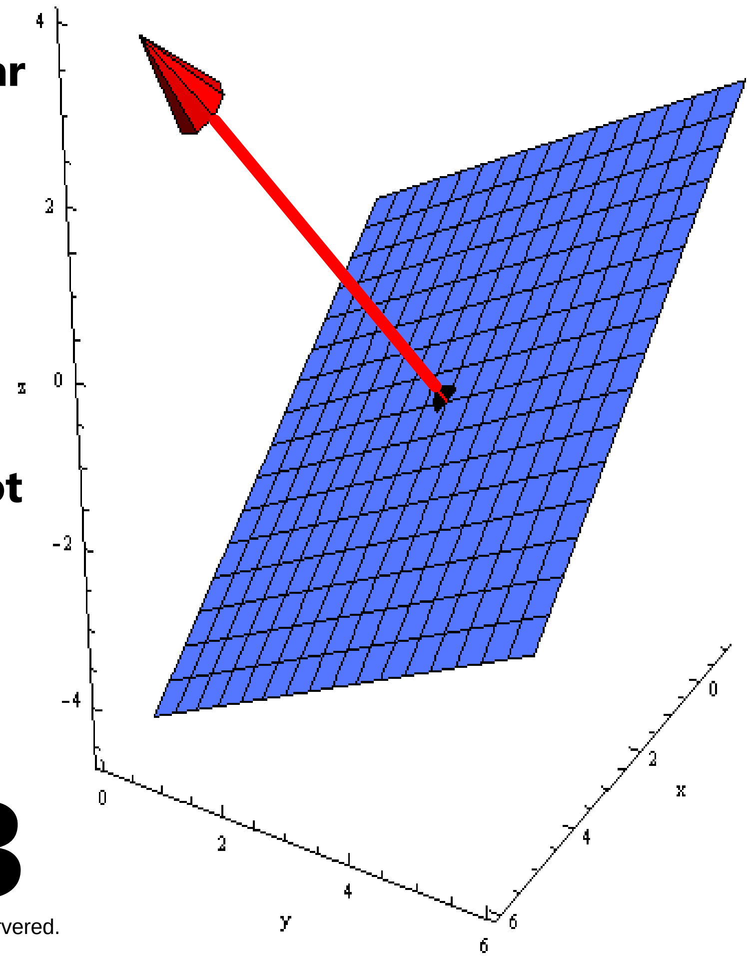
$$(x,y,z)-(2,3,-1)$$

Since $(x,y,z)-(2,3,-1)$ and $V=(4,-2,5)$ are $\perp$, their dot product is zero:

$$\big((x,y,z)-(2,3,-1)\big)\bullet(4,-2,5)=0$$

$$4(x-2)-2(y-3)+5(z+1)=0$$

$$4x-2y+5z=-3$$



Summary: Equation for a Plane

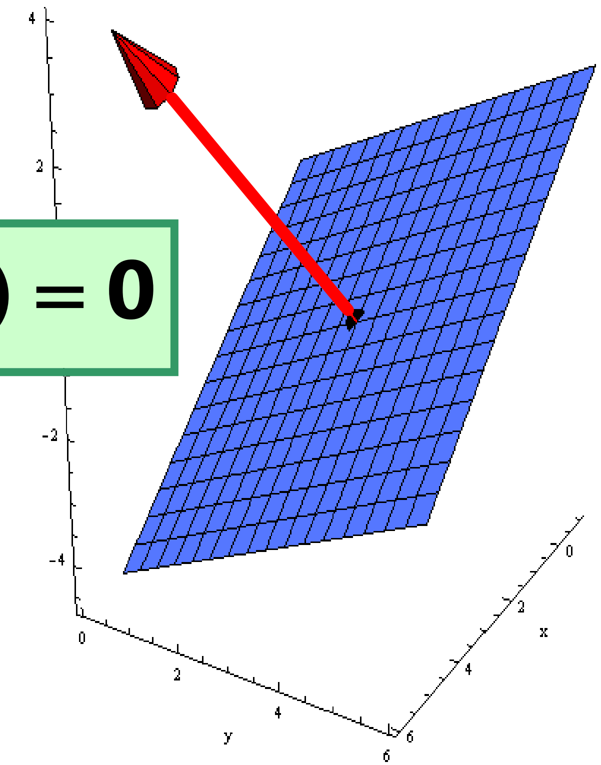
Let $P = (a, b, c)$ be a point on the plane and let $V = (v_1, v_2, v_3)$ be a normal vector to the plane.

Then:

$$\big((x, y, z) - P\big) \bullet V = 0$$

$$\boxed{v_1(x - a) + v_2(y - b) + v_3(z - c) = 0}$$

(We usually like to rewrite in the form $Ax+By+Cz=D$)

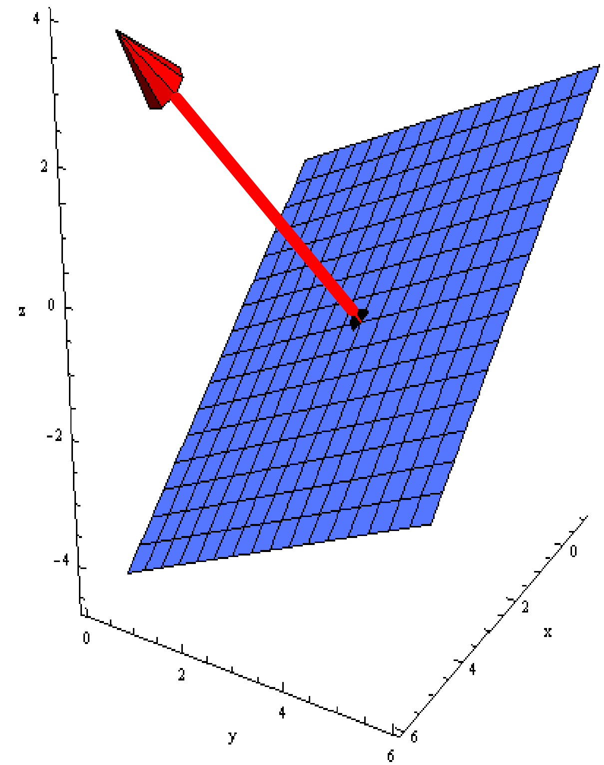


Summary: Equation for a Plane

Given an equation for a plane, you can generate a vector normal to the plane very easily:

$$v_1(x-a)+v_2(y-b)+v_3(z-c)=0$$

$$\Rightarrow (v_1, v_2, v_3) \text{ is a normal vector to the plane}$$



$$Ax+By+Cz=D$$

$$\Rightarrow (A, B, C) \text{ is a normal vector to the plane}$$

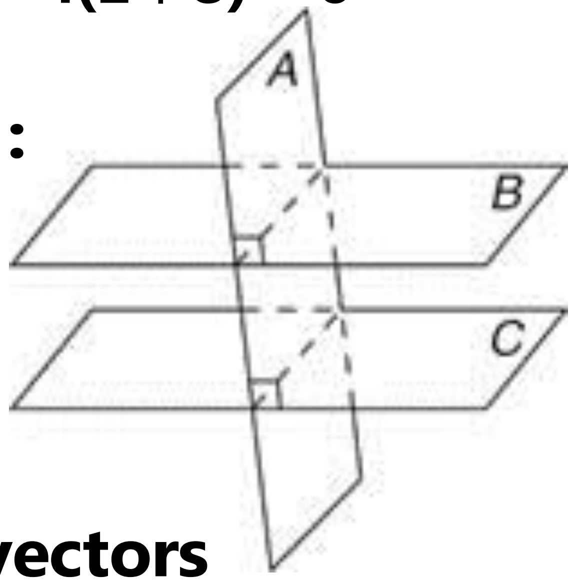
Example 2: Perpendicular Planes

Given the following two planes, show that they are perpendicular:

$$A: \ 6(x+2)-5(y+11)+2(z-1)=0$$

$$B: \ 3(x-4)+2(y-1)-4(z+8)=0$$

Ideas??



Yes, their normal vectors should be $\perp$:

$$(6,-5,2)\bullet(3,2,-4)=18-10-8$$

$$=0$$

Since the dot product of their normal vectors is 0, Plane A and Plane B are perpendicular.

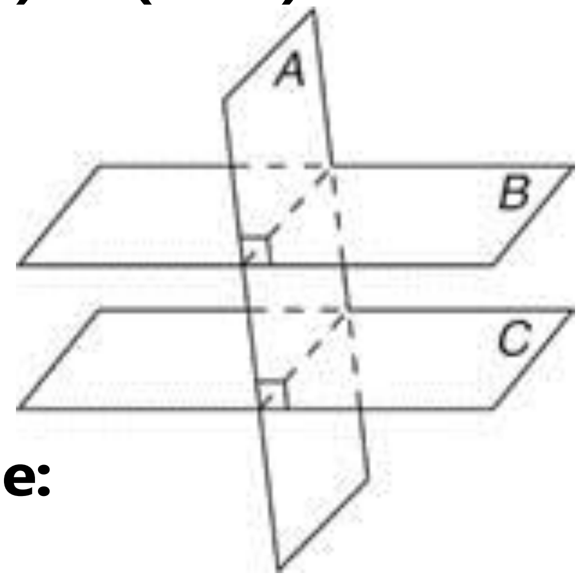
Example 3: Parallel Planes

Given the following two planes, show that they are parallel:

$$B:\ 3(x-4)+2(y-1)-4(z+8)=0$$

$$C:\ -6(x+3)-4(y+1)+8(z-7)=0$$

Ideas??



First, show their normal vectors are multiples of each other:

$$(-6,-4,8)=-2(3,2,-4)$$

Next, make sure they aren't the same plane:

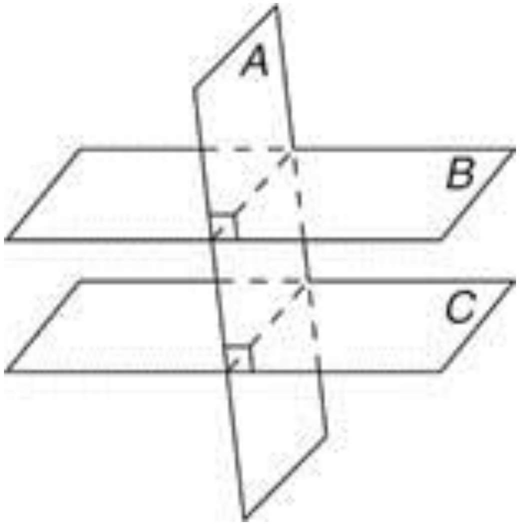
$$(4,1,-8) \text{ should not satisfy } C: -6(x+3)-4(y+1)+8(z-7)=0:$$

$$-6(4+3)-4(1+1)+8(-8-7)=-170$$

Therefore Plane B and Plane C are parallel planes.

Summary: Parallel and Perpendicular Planes

Given two planes with equations $\begin{cases} Ax + By + Cz = D \\ Ex + Fy + Gz = H \end{cases}$, we can determine if they are parallel or perpendicular as follows:



$\underline{\text{Perpendicular}}: (A,B,C) \bullet (E,F,G) = 0$

$\underline{\text{Parallel}}: \exists\, k \in \mathbb{R} \text{ s.t. } (A,B,C) = k(E,F,G)$
and $\exists\, P \in Plane_1 \text{ s.t. } P \notin Plane_2$

$\underline{\text{Same Plane}}: \exists\, k \in \mathbb{R} \text{ s.t. } (A,B,C) = k(E,F,G)$
and $\exists\, P \in Plane_1 \text{ s.t. } P \in Plane_2$

Warning: Lines and Planes Behave Differently!

- Describing the intersection between two planes is a lot easier than the intersection between two lines in 3 dimensions!
- Here's the main hiccup to watch out for: two lines in three dimensions do NOT necessarily intersect (they can be skew)
- Let's look at a few different examples!

Are the Following Pairs of LINES Perpendicular?

$L(t) = (1,4,2) + t(3,1,1)$
$M(t) = (1,4,2) + t(0,-1,1)$

L(t) and M(t) intersect at (1,4,2) and $(3,1,1) \bullet (0,-1,1) = 0$, so these lines are $\perp$

$L(t) = (2,0,-1) + t(3,1,1)$
$M(t) = (5,1,0) + t(0,-1,1)$

The lines intersect at $L(1) = M(0) = (5,1,0)$ and $(3,1,1) \bullet (0,-1,1) = 0$, so these lines are $\perp$

$L(t) = (5,1,0) + t(3,1,1)$
$M(t) = (-3,1,2) + t(0,-1,1)$

L(t) and M(t) do not intersect, so these lines are not $\perp$ (they are skew)

Are the Following Pairs of LINES Parallel?

$L(t) = (1,4,2) + t(3,1,1)$
$M(t) = (5,9,6) + t(3,1,1)$

$(1,4,2) \notin M(t)$ and the lines have the same generating vector, so they are $\parallel$

$L(t) = (1,4,2) + t(3,1,1)$
$M(t) = (-2,3,1) + t(3,1,1)$

$M(1) = (1,4,2) \in L(t)$ and the lines have the same generating vector, so they are the same line twice.

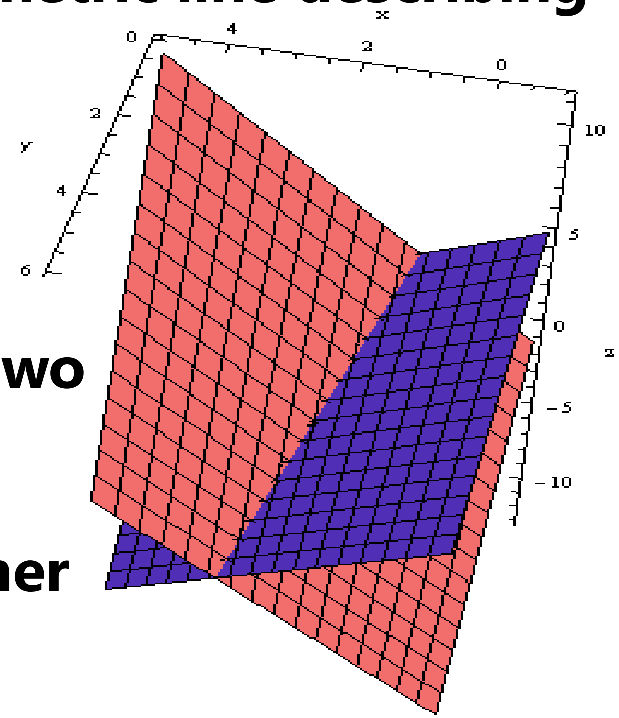
$L(t) = (1,4,2) + t(3,1,1)$
$M(t) = (10,7,5) + t(6,2,2)$

$L(3) = (10,7,5) \in M(t)$ and the generating vectors of the lines are multiples of each other, so they are the same line twice.

Example 4: The Line of Intersection

Given the following two planes, find the parametric line describing their intersection:

$$\begin{aligned} A:&\quad 3x - 4y + z = 5 \\ B:&\quad 2x + 4y - 2z = -3 \end{aligned}$$



Eliminate one variable:

$$5x \quad - z = 2$$

So $z = 5x - 2$. We can use this relation to find two points, P and Q on the line of intersection:

When $x = 1$, $z = 5(1) - 2 = 3$. Plug back into either plane to get that $y = \frac{1}{4}$. So $P = \left(1, \frac{1}{4}, 3\right)$

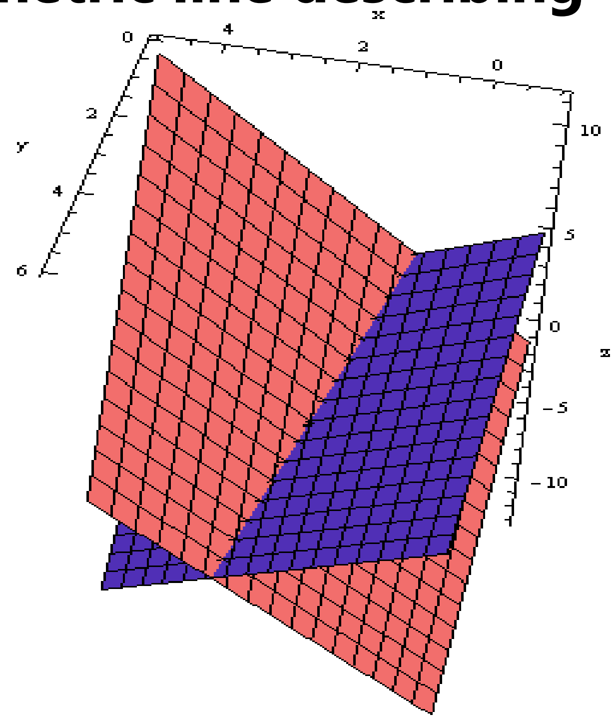
When $x = -1$, $z = 5(-1) - 2 = -7$. Plug back into either plane to get that $y = -\frac{15}{4}$. So $Q = \left(-1, -\frac{15}{4}, -7\right)$

Example 4: The Line of Intersection

Given the following two planes, find the parametric line describing their intersection:

$$A:\ 3x - 4y + z = 5$$

$$B:\ 2x + 4y - 2z = -3$$



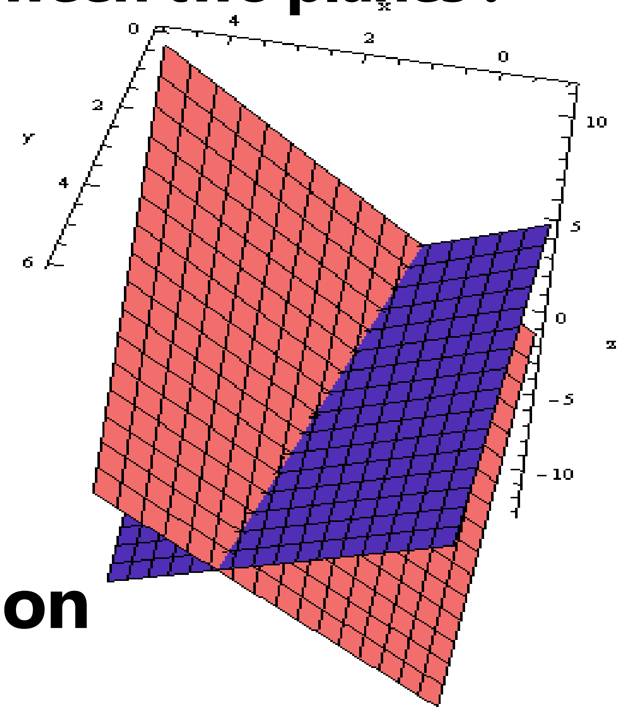
So $P = \left(1, \frac{1}{4}, 3\right)$ and $Q = \left(-1, -\frac{15}{4}, -7\right)$ lie on BOTH planes :

$$L(t) = P + t(Q - P)$$

$$L(t) = \left(1, \frac{1}{4}, 3\right) + t(-2, -4, -10) \qquad t \in (-\infty, \infty)$$

Summary: The Line of Intersection

To find the parametric line of intersection between two planes :



1) Eliminate a variable between the two equations to find a relation between 2 variables

2) Find two points on the line of intersection using this relation

3) Use these two points in $L(t) = P + t(Q - P)$

Example 5: Plane from Three Points

Given the following set of three noncollinear points, find the parametric equation of the plane passing through them:

$$P = (5,1,-6)$$
$$Q = (3,-1,4)$$
$$R = (1,2,4)$$

Parametrically this is easy :

For a line, we just have a point P and a generating vector V:

$$L(t) = P + tV \qquad t \in (-\infty,\infty)$$

For a plane, we just need a point P and two generating vectors, V and W, with different directions (linearly independent vectors):

$$M(s,t) = P + tV + sW \qquad s,t \in (-\infty,\infty)$$

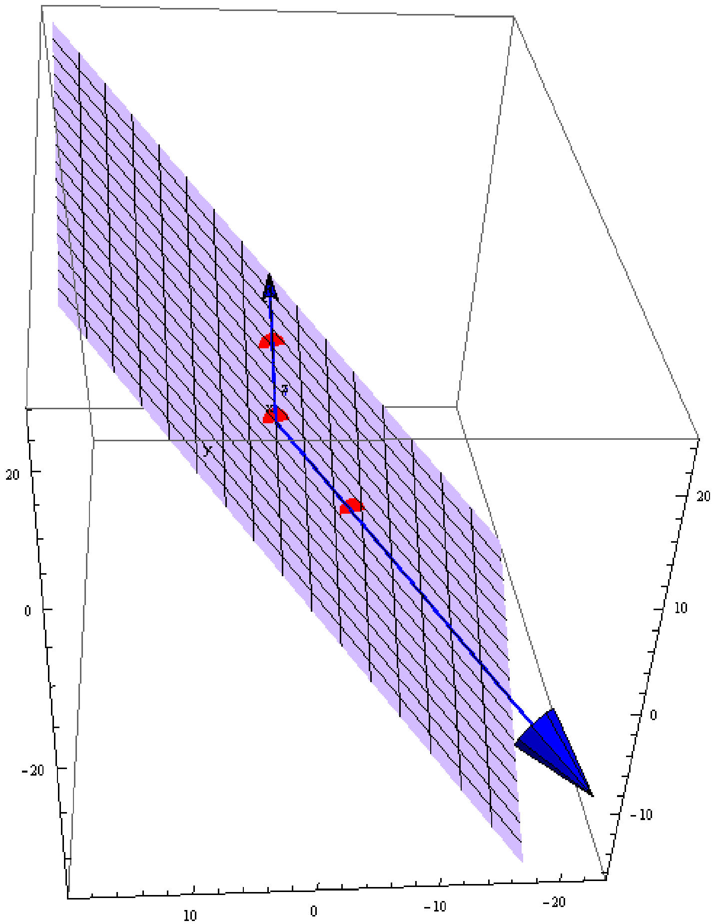
$$\begin{aligned} V &= Q - P \\ &= (3,-1,4) - (5,1,-6) \\ &= (-2,-2,10) \end{aligned}$$

$$\begin{aligned} W &= R - P \\ &= (1,2,4) - (5,1,-6) \\ &= (-4,1,10) \end{aligned}$$

$$s,t \in (-\infty,\infty)$$

$$M(s,t) = (5,1,-6) + t(-2,-2,10) + s(-4,1,10)$$

Example 5: Plane from Three Points



$s, t \in (-\infty, \infty)$

$$M(s,t) = (5,1,-6) + t(-2,-2,10) + s(-4,1,10)$$

Example 5: Plane from Three Points

Given the following set of three noncollinear points, find the xyz-equation of the plane passing through them:

$$P = (5, 1, -6)$$
$$Q = (3, -1, 4)$$
$$R = (1, 2, 4)$$

You could plug each point into Ax+By+Cz=D and solve a system... but that's no fun!

If we could find a normal vector to the surface, we could use:

$$v_1(x-a)+v_2(y-b)+v_3(z-c)=0$$

<u>Task</u> : Find a vector perpendicular to both

$$V=(-2,-2,10) \text{ and } W=(-4,1,10)$$

Review: Determinant of a 2 x 2 Matrix

- **Every square (n x n) matrix has a real number associated to it called the *determinant***

$$\det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

- **Short form:**

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

Review: Determinant of a 3 x 3 Matrix

- Method: Expansion by Minors

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\underbrace{\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}}_{M_{1,1}} - a_{1,2}\underbrace{\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix}}_{M_{1,2}} + a_{1,3}\underbrace{\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}}_{M_{1,3}}$$

These determinants are the "minors" of the original matrix.

Example 5: Plane from Three Points

Given the following set of three noncollinear points, find the xyz-equation of the plane passing through them:

$$P = (5, 1, -6)$$
$$Q = (3, -1, 4)$$
$$R = (1, 2, 4)$$

You could plug each point into Ax+By+Cz=D and solve a system... but that's no fun!

If we could find a normal vector to the surface, we could use:

$$v_1(x-a)+v_2(y-b)+v_3(z-c)=0$$

<u>Task :</u> Find a vector perpendicular to both

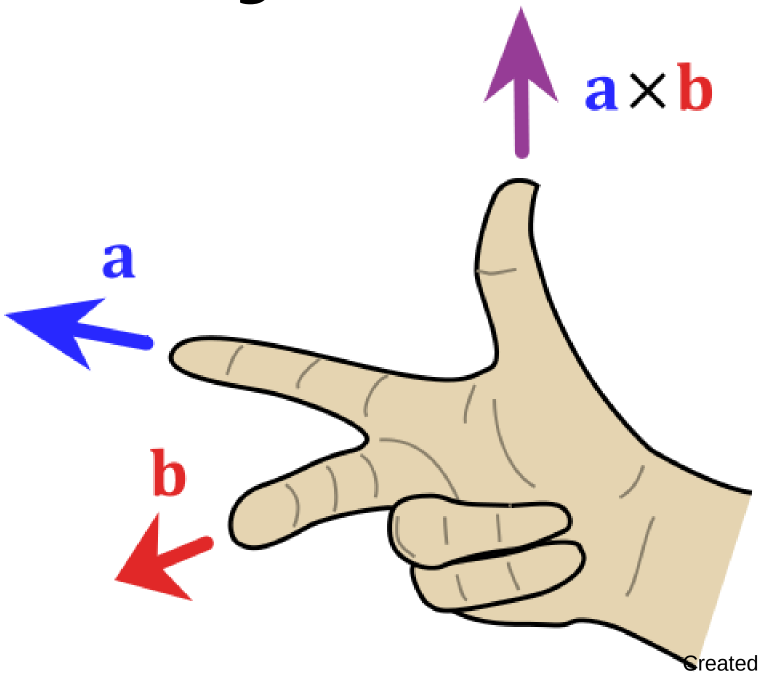
$$V = (-2, -2, 10) \text{ and } W = (-4, 1, 10)$$

Defining the Cross Product

We will introduce a new binary operation called the cross product.

The cross product should be a way to take two vectors, **a** and **b**, and find a third, $\mathbf{a} \times \mathbf{b}$, that is perpendicular to both.

For the right hand rule with $\mathbf{a} \times \mathbf{b}$, the first vector is always your index finger and the second vector is always your middle finger!

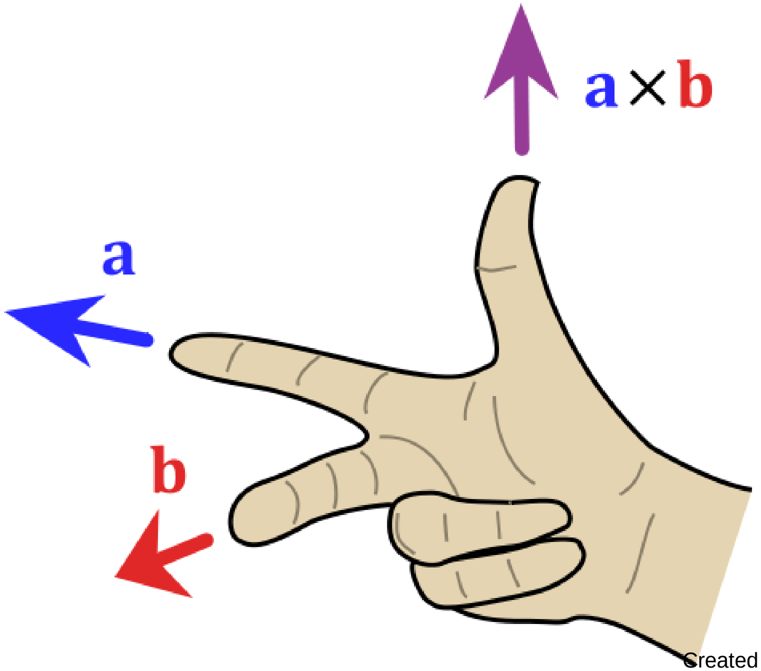


Defining the Cross Product

For the right hand rule with $a \times b$, the first vector is always your index finger and the second vector is always your middle finger!

Lay two pens on the table and call them a and b.

Find $a \times b$. How does it relate to $b \times a$?

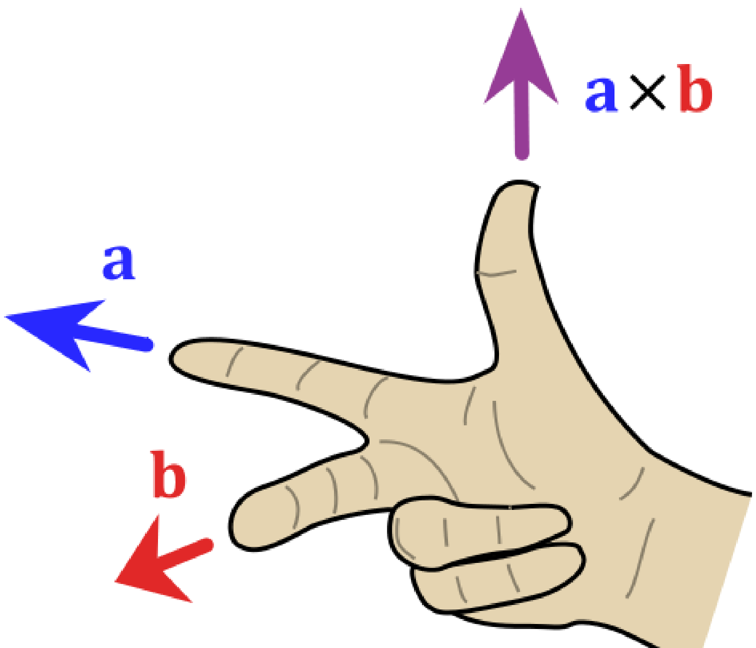


$$b \times a = -(a \times b)$$

Defining the Cross Product

For the right hand rule with a × b, the first vector is always your index finger and the second vector is always your middle finger!

Lay two pens on the table in the same spot and call them a and a. Find a × a. How does it relate to a × a?



$$\mathbf{a} \times \mathbf{a} = (0,0,0)$$

$$= \vec{0}$$

Defining the Cross Product

Let $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$. These three vectors are called the standard a set of unit basis vectors for xyz-space.

They are unit vectors since $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$

They are basis vectors since they point in three different directions (so any vector can be described as sums/multiples of these vectors)

Ex. $(11,-21,16)$ can be written as $11\mathbf{i} - 21\mathbf{j} + 16\mathbf{k}$

Ex. $(5,0,1)$ can be written as $5\mathbf{i} + \mathbf{k}$

They are standard basis vectors since they only have x-, y-, or z-components

Defining the Cross Product

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$

$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

$\mathbf{i} \times \mathbf{i} = \vec{0}$

$\mathbf{j} \times \mathbf{k} = \mathbf{i}$

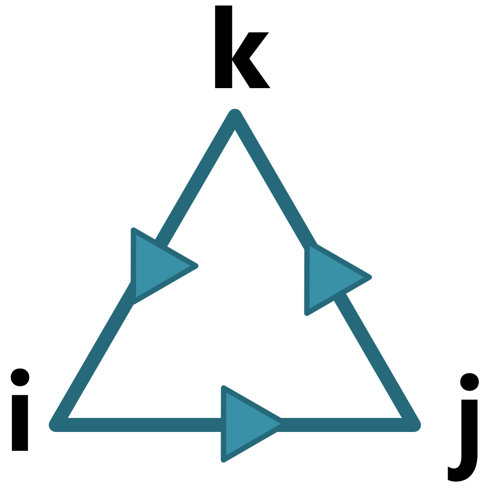
$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

$\mathbf{j} \times \mathbf{j} = \vec{0}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

$\mathbf{k} \times \mathbf{k} = \vec{0}$



Calculate $V \times W$ for $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$:

$$V \times W = (v_1 i + v_2 j + v_3 k) \times (w_1 i + w_2 j + w_3 k)$$

$$= v_1 w_1 i \times i + v_1 w_2 i \times j + v_1 w_3 i \times k +$$
$$v_2 w_1 j \times i + v_2 w_2 j \times j + v_2 w_3 j \times k +$$
$$v_3 w_1 k \times i + v_3 w_2 k \times j + v_3 w_3 k \times k$$

$$= v_1 w_1 (\vec{0}) + v_1 w_2 (k) + v_1 w_3 (-j) +$$
$$v_2 w_1 (-k) + v_2 w_2 (\vec{0}) + v_2 w_3 (i) +$$
$$v_3 w_1 (j) + v_3 w_2 (-i) + v_3 w_3 (\vec{0})$$

$$= (v_2 w_3 - v_3 w_2) i + (v_3 w_1 - v_1 w_3) j + (v_1 w_2 - v_2 w_1) k$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} k = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$i \times j = k$

$j \times k = i$

$k \times i = j$

$j \times i = -k$

$k \times j = -i$

$i \times k = -j$

$i \times i = \vec{0}$

$j \times j = \vec{0}$

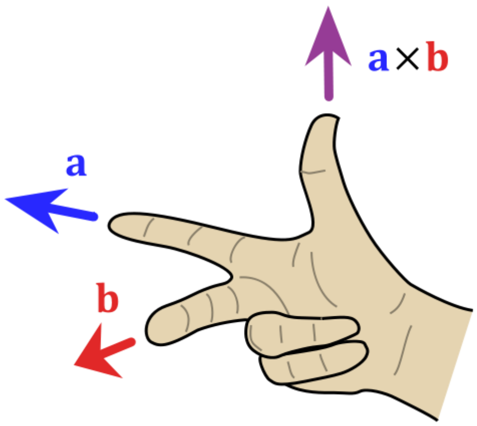
$k \times k = \vec{0}$

Summary: The Cross Product

Calculate $V \times W$ for $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$:

$$V \times W = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Remember that $V \times W$ generates a vector perpendicular to both V and W!



Use Mathematica to show

$V \bullet (V \times W) = 0$ and $W \bullet (V \times W) = 0$!

Verifying the Cross Product

We can verify that the cross product gives us a perpendicular vector to V and W for any choices of V and W. Try this in Mathematica:

```
V={v1,v2,v3};
W={w1,w2,w3};
VxW=Cross[V,W];
Together[V.VxW]
Together[W.VxW]
```

Example 6: The Cross Product

Calculate $(-2,-2,10)\times(-4,1,10)$:

$$(-2,-2,10)\times(-4,1,10)=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 10 \\ -4 & 1 & 10 \end{vmatrix}$$

$$=\mathbf{i}\begin{vmatrix} -2 & 10 \\ 1 & 10 \end{vmatrix}-\mathbf{j}\begin{vmatrix} -2 & 10 \\ -4 & 10 \end{vmatrix}+\mathbf{k}\begin{vmatrix} -2 & -2 \\ -4 & 1 \end{vmatrix}$$

$$=-30\mathbf{i}-20\mathbf{j}-10\mathbf{k}$$

$$=(-30,-20,-10)$$

Example 5: Plane from Three Points (CONTINUED)

Given the following set of three noncollinear points, find the xyz-equation of the plane passing through them:

$P = (5,1,-6)$

$Q = (3,-1,4)$

$R = (1,2,4)$

Now that we have the cross product, we know that $V \times W$ is a normal vector to the plane!

$$V \times W = (-2,-2,10) \times (-4,1,10)$$

$$= (-30,-20,-10) \quad \text{(from previous slide!)}$$

$$v_1(x-a) + v_2(y-b) + v_3(z-c) = 0$$

$$-30(x-5) - 20(y-1) - 10(z+6) = 0$$

$$3(x-5) + 2(y-1) + (z+6) = 0$$

$$3x + 2y + z = 11$$

Example 6: The Magnitude of the Cross Product

Let $V = (6,-2,5)$ and $W = (-8,1,10)$. Find $\left|V \times W\right|$.

You COULD do this the long way:

$$(6,-2,5)\times(-8,1,10) = \begin{vmatrix} i & j & k \\ 6 & -2 & 5 \\ -8 & 1 & 10 \end{vmatrix}$$

$$= i\begin{vmatrix} -2 & 5 \\ 1 & 10 \end{vmatrix} - j\begin{vmatrix} 6 & 5 \\ -8 & 10 \end{vmatrix} + k\begin{vmatrix} 6 & -2 \\ -8 & 1 \end{vmatrix}$$

$$= -25i - 100j - 10k$$

$$= (-25,-100,-10)$$

$$\left|V \times W\right| = \left|(-25,-100,-10)\right|$$

$$= \sqrt{(-25,-100,-10)\bullet(-25,-100,-10)}$$

$$= 5\sqrt{429}$$

Example 6 : The Magnitude of the Cross Product

Shortcut : $|V \times W| = |V||W|\sin(\theta)$ **where θ is the angle between V and W.**

Proof : YOU will prove this in your homework. You MUST know its geometric significance before you take the chapter quiz.

Note : Memorize this formula. Study it. Use it. Don't forget it. You will need it along with the analogous formula for dot products:

$$V \bullet W = |V||W|\cos(\theta)$$

Example 6 : The Magnitude of the Cross Product

Let $V = (6, -2, 5)$ and $W = (-8, 1, 10)$. Find $|V \times W|$.

Now the short way:

$(6, -2, 5) \bullet (-8, 1, 10) = -48 - 2 + 50 = 0.$

So the angle between V and W is $\theta =$ Answer Under the Box!

$$|V \times W| = |V||W|\sin(\theta)$$

$$= \sqrt{65}\sqrt{165}\sin\left(? \right)$$

$$= 5\sqrt{429}$$