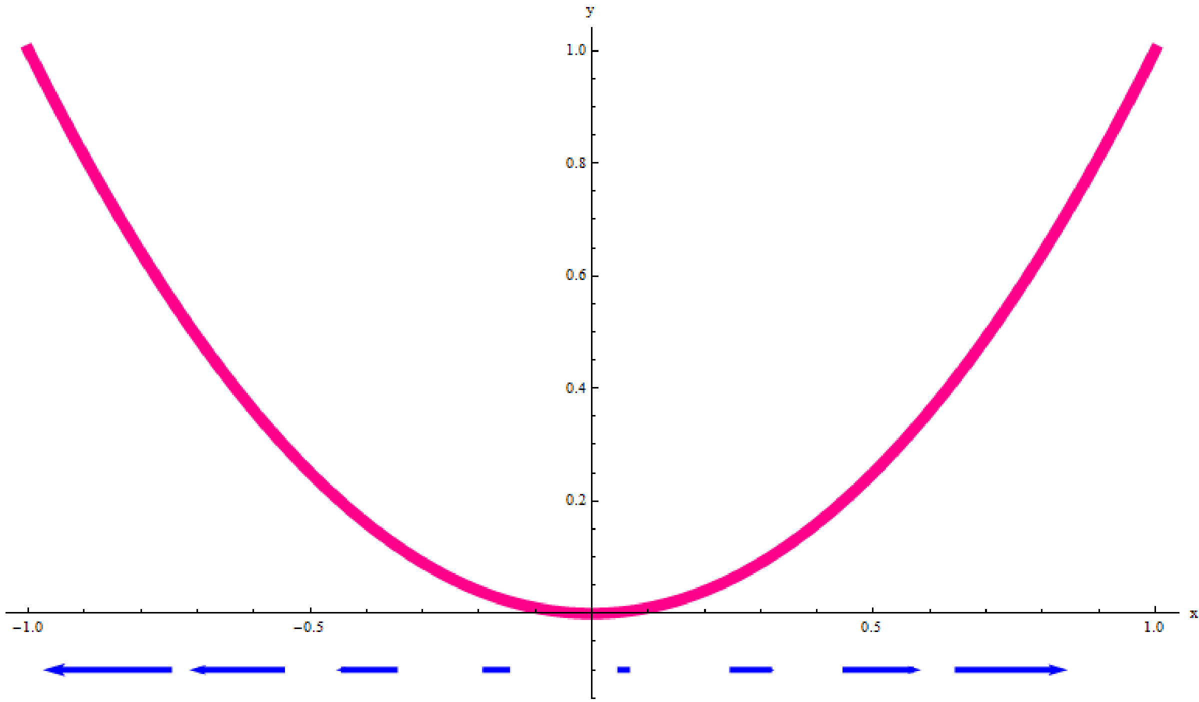
Lesson 4:

Gradient Vectors, Level Curves, Maximums/Minimums/Saddle Points

Example 1: The Gradient Vector

Let $f(x) = x^2$. Then $\frac{df}{dx} = 2x$. This can be thought of as a vector that tells you the direction of greatest initial increase on the curve. The magnitude of the vector tells you how steep the increase.

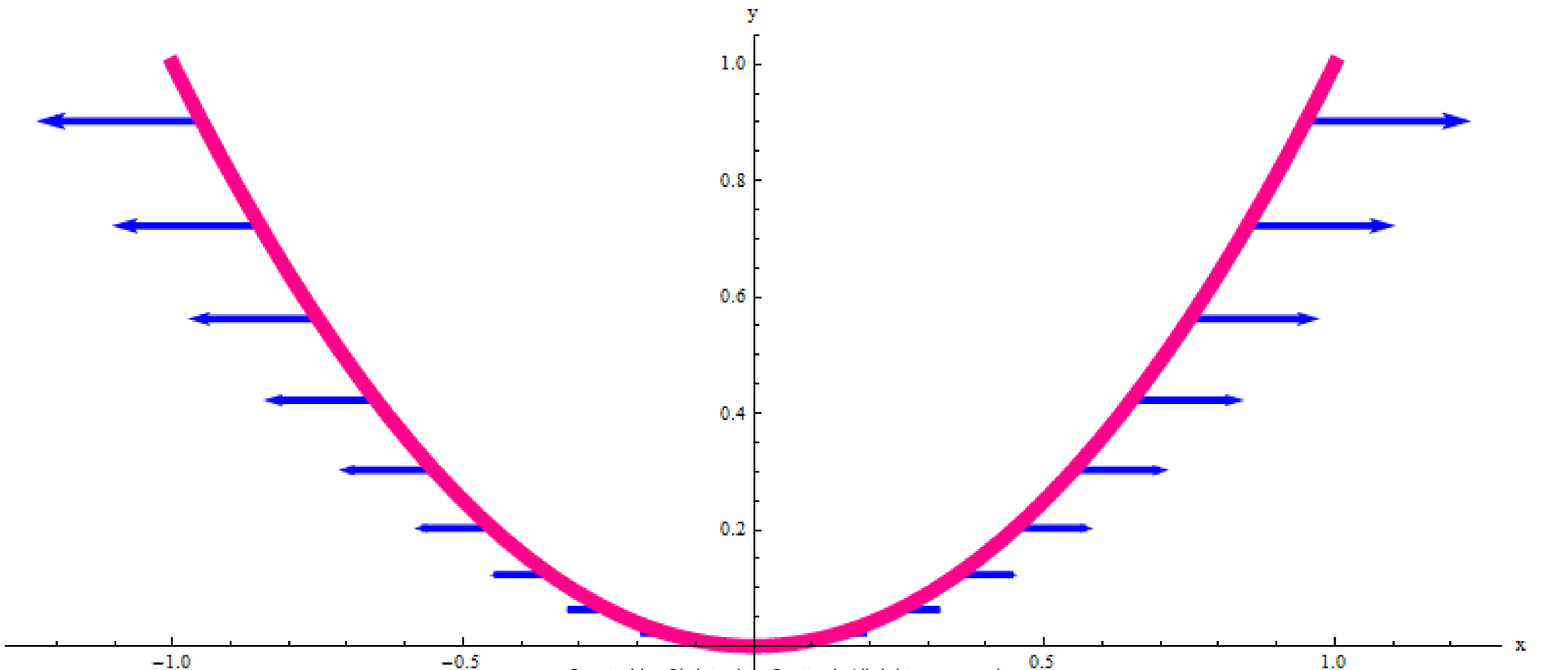
Let's try a few x-values in $\frac{df}{dx}$:



Example 1: The Gradient Vector

We can put the tails of our vectors on the curve itself to get picture that's a little easier to work with:

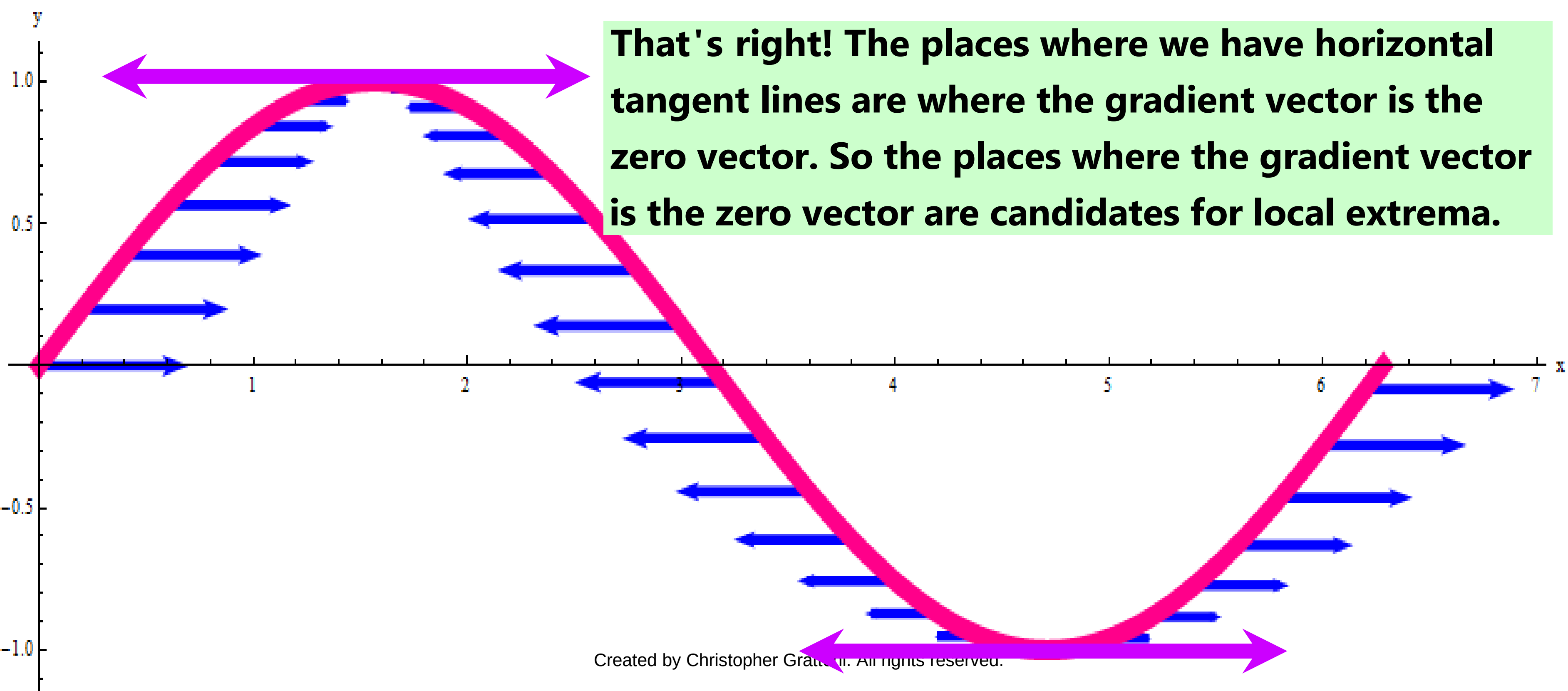
What do you notice about the magnitude of the gradient vector at x=0?



Example 2: The Gradient Vector

We can try again with $f(x) = \sin(x)$:

What do you notice about the magnitude of the gradient vector at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$?



That's right! The places where we have horizontal tangent lines are where the gradient vector is the zero vector. So the places where the gradient vector is the zero vector are candidates for local extrema.

Definition: The Gradient Vector

Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables. Then the gradient vector is defined as follows:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

The gradient vector is designed to point in the direction of the greatest INITIAL increase on your curve/surface/etc.

Notice that the gradient vector always lives in one dimension lower than function does. Surface in 3D space? 2D gradient vector. Curve in 2D space? 1D gradient vector. Hypersurface in 4D? 3D gradient vector.

Definition: The Gradient Vector

Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables and let

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Ex. Find ∇f given $f(x) = \sin(x)$.

$$\nabla f(x) = \left(\cos(x) \right)$$

Ex. If $f(x, y) = x^3 + y^3 - 3x - 3y$, find ∇f.

$$\nabla f(x, y) = \left(3x^2 - 3, 3y^2 - 3 \right)$$

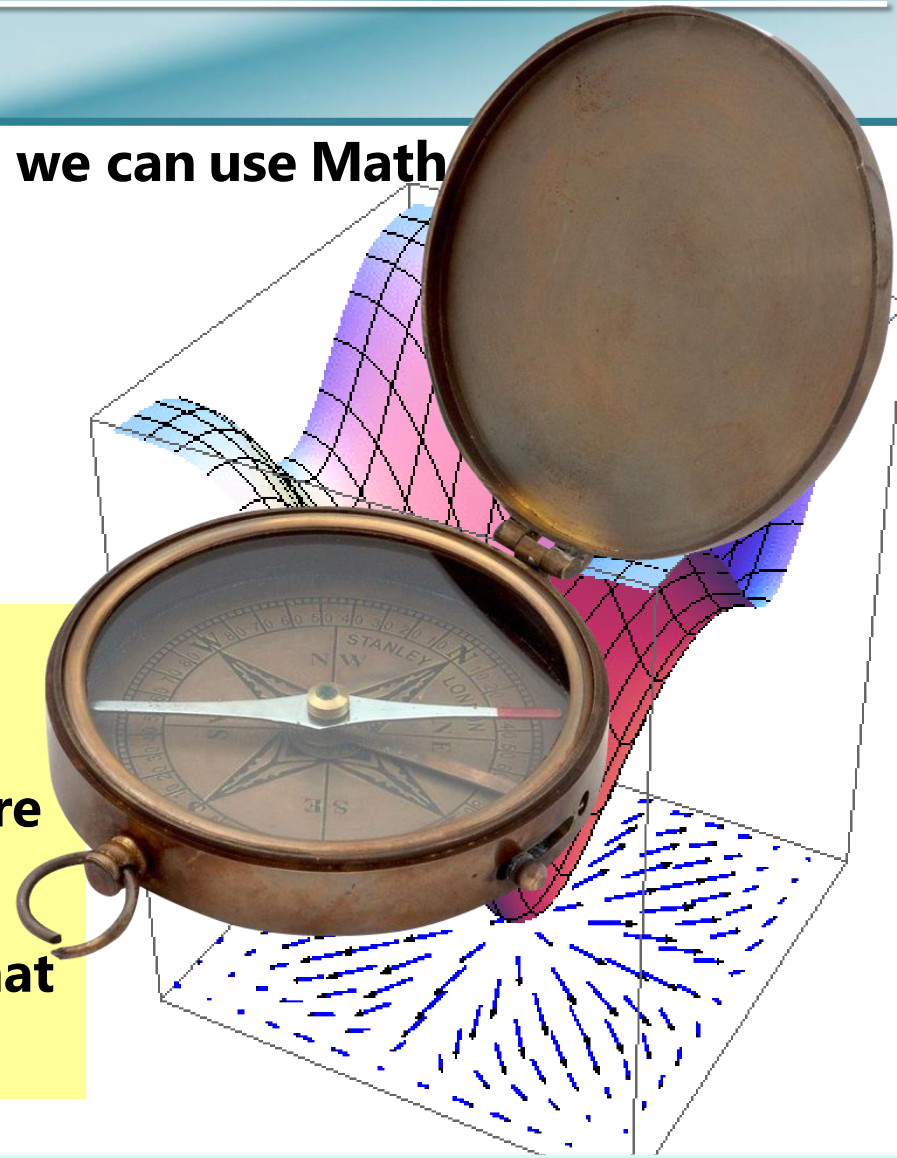
Theorem: If $f(x_1, x_2, \ldots, x_n)$ is a function of n variables, then the candidates for local maxima/minima are where $\nabla f = (0, 0, \ldots, 0)$ or where ∇f is undefined.

NOTE : These are just CANDIDATES, not necessarily extrema!!

Example 3: A Surface and Gradient Field

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$. Then we can use Math

to find $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.



This is the surface plotted together with a projected gradient field below it.

The surface is like a mountain and the gradient vector is like a magic compass that, no matter where you are standing on the mountain, points in the immediate direction that is the steepest uphill.

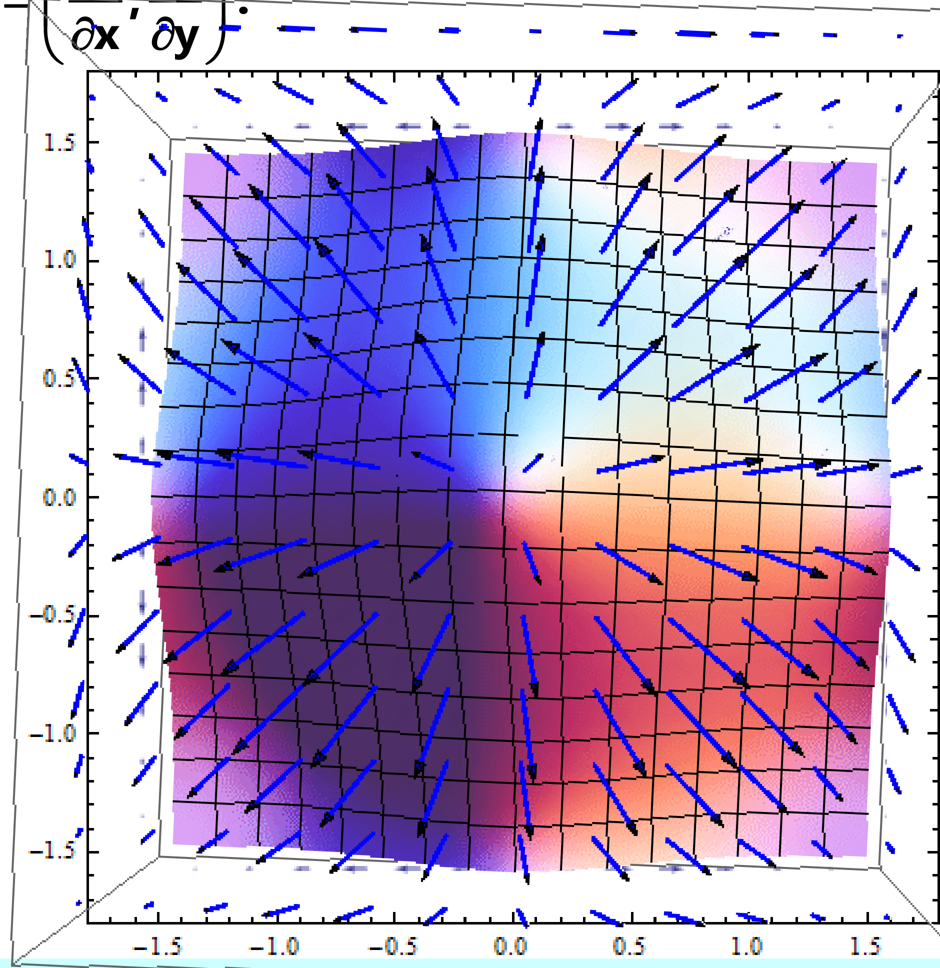
$$\nabla f(x,y) = \left(4\cos^3(x)\sin(x) + 4\cos(x)\sin(x)\cos(y), 2\cos^2(x)\sin(y) + 2\cos(y)\sin(y)\right)$$

Example 3: A Surface and Gradient Field

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$:

One of the main reasons we like to plot the gradient vectors (called a gradient <u>vector field</u>) is that we can figure out quite a bit about a 3D surface without the hard work of a 3D plot:

The vectors tell you the direction of greatest initial increase on the surface at a given point, and their magnitude tells you how steep it is!

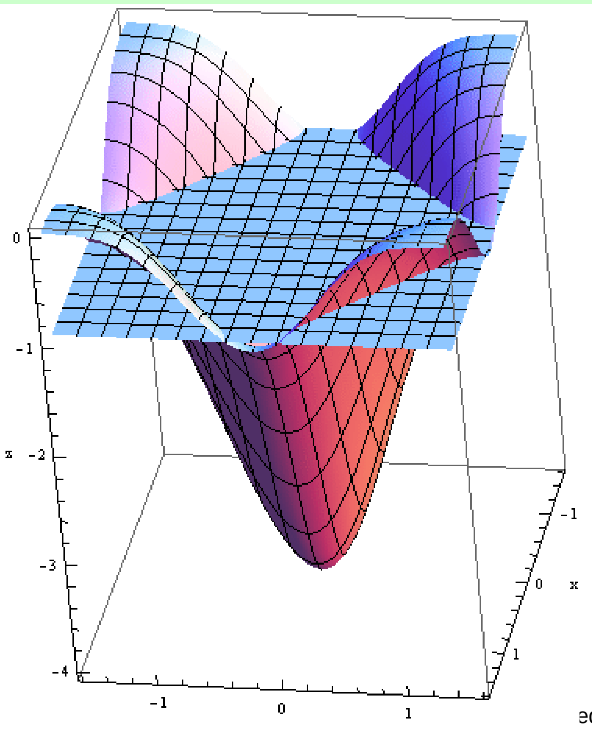


$$\nabla f(x,y) = \left(4\cos^3(x)\sin(x) + 4\cos(x)\sin(x)\cos(y), 2\cos^2(x)\sin(y) + 2\cos(y)\sin(y)\right)$$

Example 4: A Surface and Contour Plot

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ **and** $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$**:**

We can also generate a "contour plot" of our surface. This is also called a set of level curves. These are merely slices of the surface projected onto the xy-plane:



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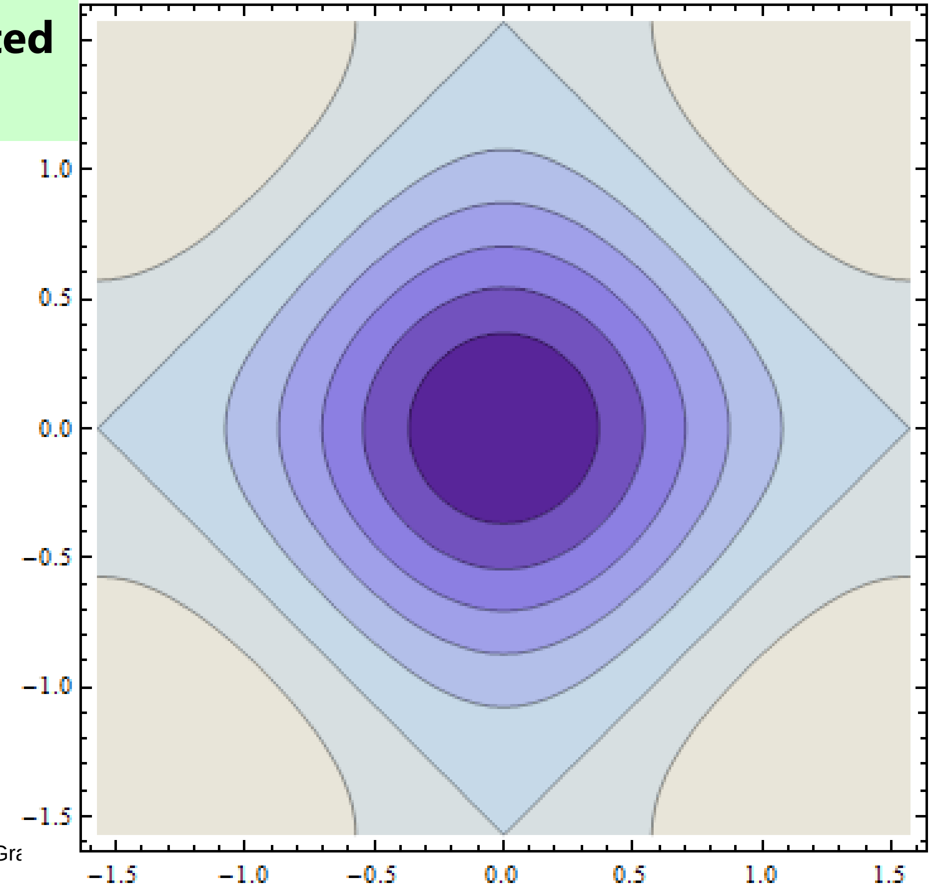


Example 4: Contour Plots in Real Life

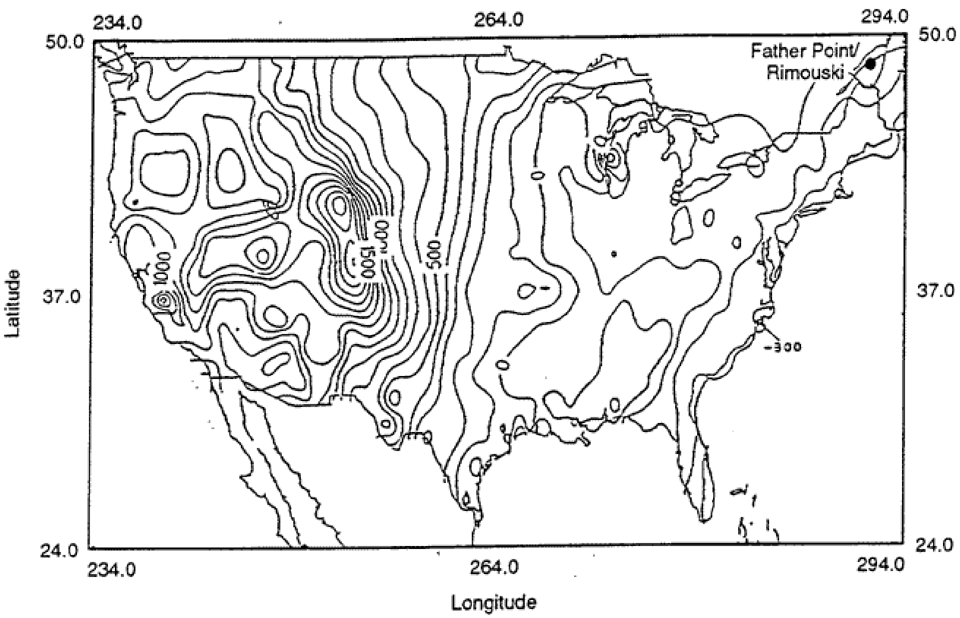


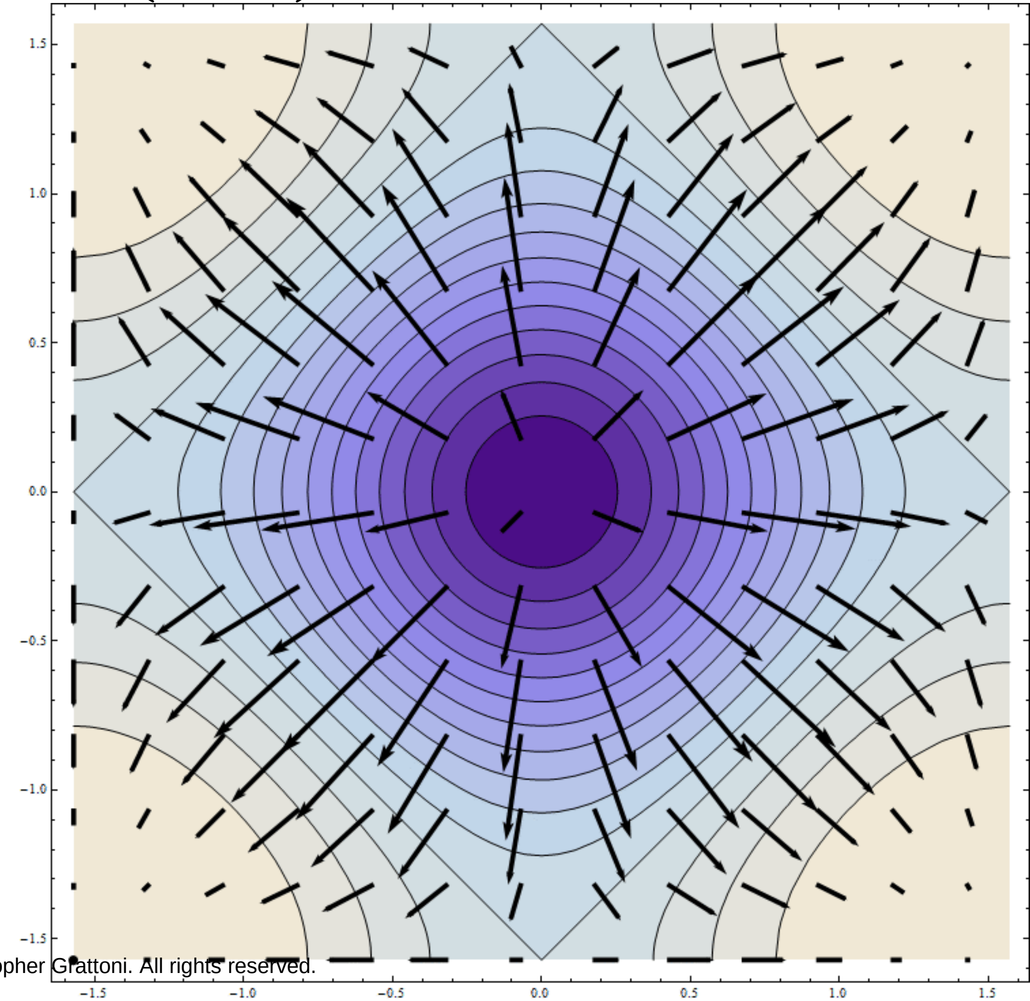
Figure 4. Contour map depicting height differences between NAVD 88 and NGVD 29 (units = mm).

This is merely just a technique for representing 3D data in 2D.

Example 4: A Surface and Contour Plot

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$:

We can put our contour plot and gradient field together:



Recall that the gradient always points in the direction of greatest initial increase, so it must get you from one level curve to the next as efficiently as possible. That is, the gradient vector is perpendicular to the level curve passing through its tail.

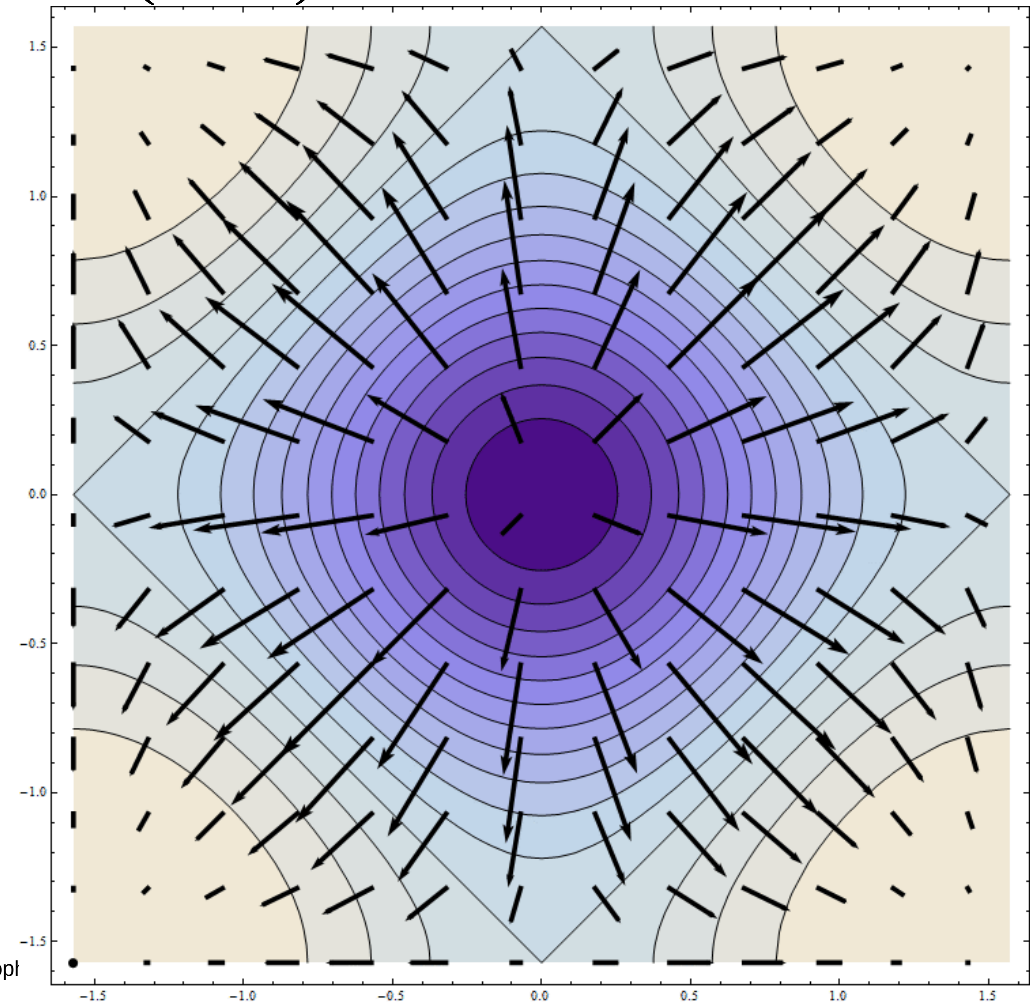
Example 4: A Surface and Contour Plot

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ **and** $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$**:**

Theorem: The gradient is always perpendicular to the level curve through its tail.

Proof: Later in the notes.

Note : The gradient is not necessarily perpendicular to the surface itself, just its 2D level curves!



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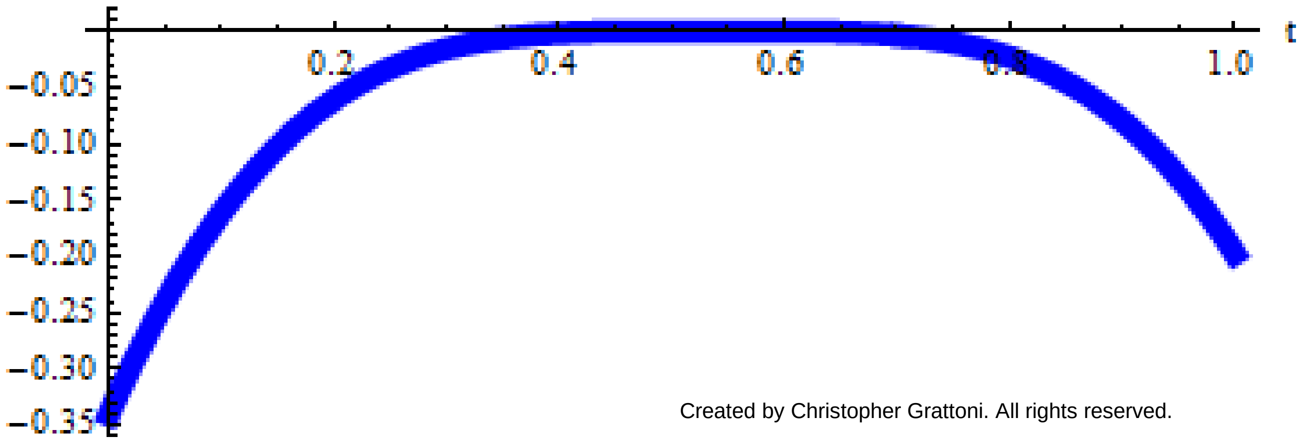
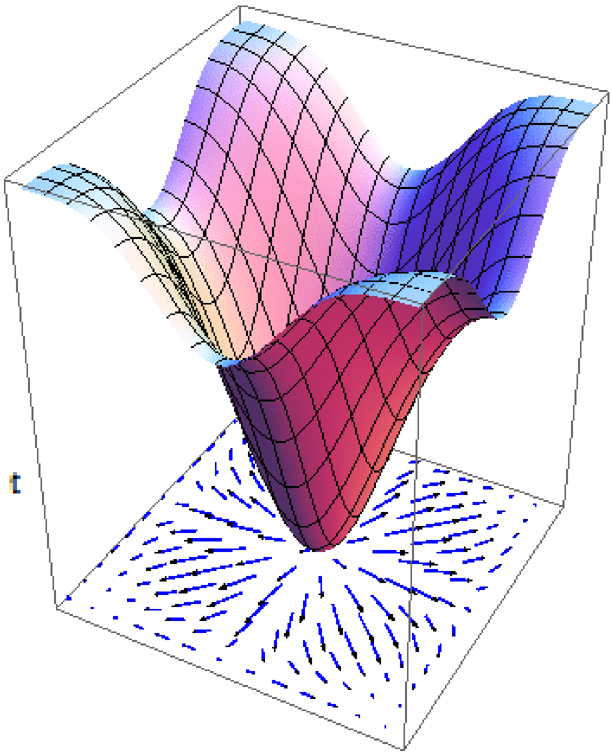
Example 5: The Gradient Points in the Direction of Greatest Initial Increase

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Show that if we are at point (1,1,f(1,1)) and move in the direction of the gradient $\nabla f(1,1)$, we must go up before we go down:

Here is a plot of f(x(t),y(t)) versus t where $(x(t),y(t)) = (1,1) + t\nabla f(1,1)$ for $0 \le t \le 1$:





Example 6: A Path Along our Surface

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and let $(x(t),y(t)) = \left(\cos^2(1+t), \sin(2t)\right)$.

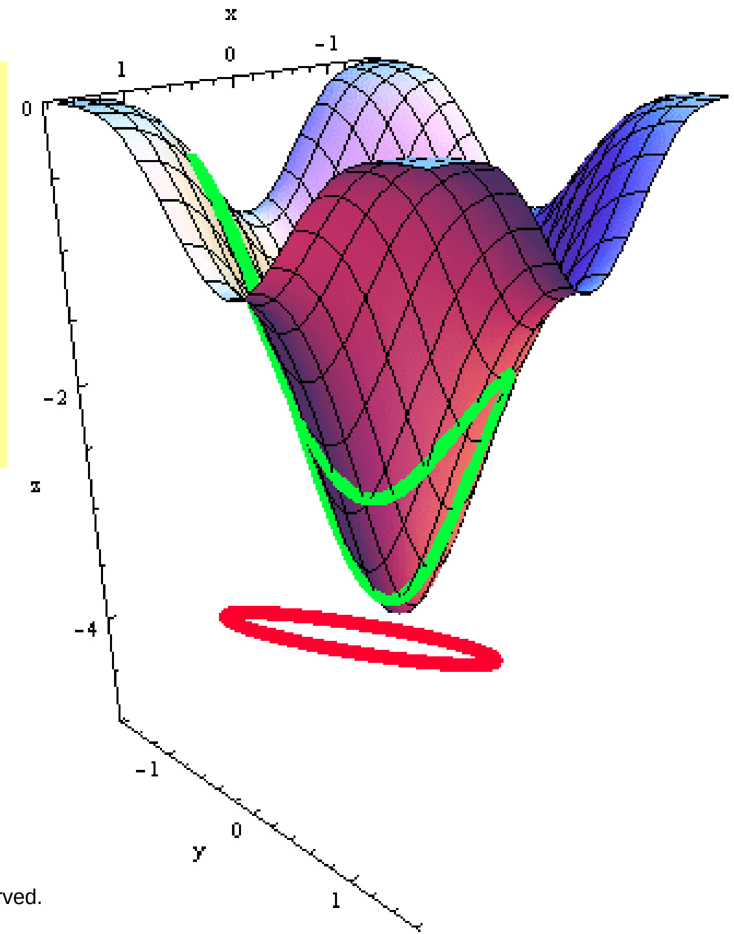
Finally, plot $(x(t), y(t), f(x(t), y(t)))$, a path on the surface.

$(x, y, f(x,y))$: the surface (a mountain)

$(x(t), y(t))$: a parametric path (map of a hiking trail)

$(x(t), y(t), f(x(t), y(t)))$: path on the surface

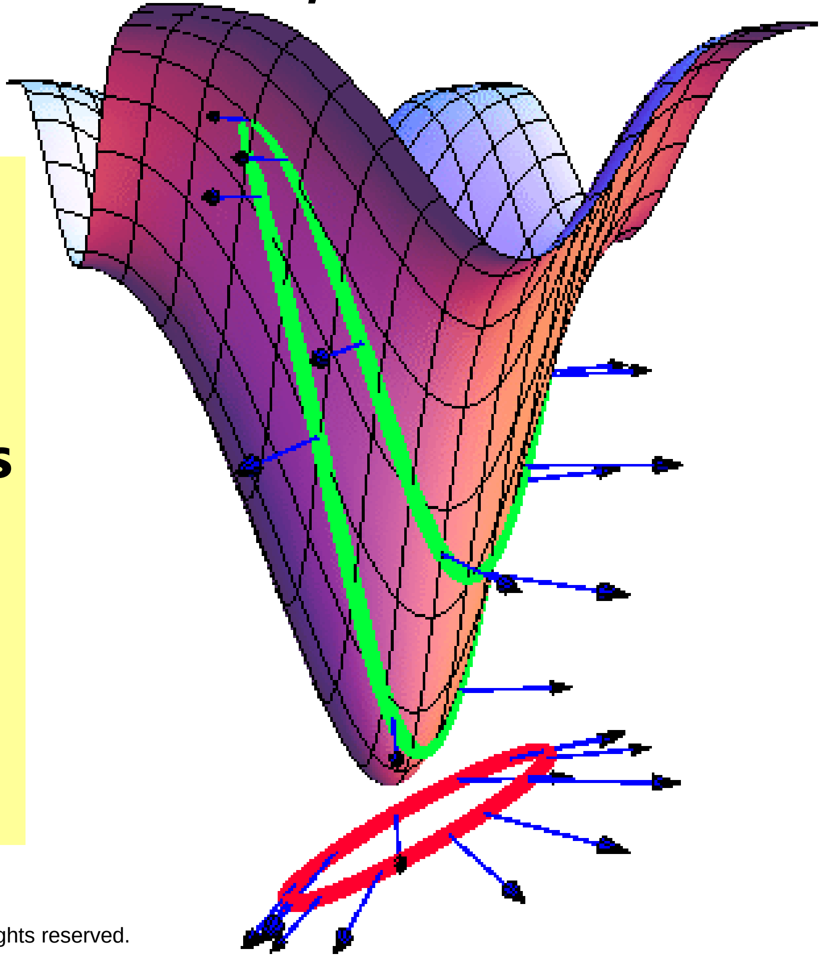
(the actual hiking trail on that mountain)



Example 6: A Path Along our Surface

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and let $(x(t), y(t)) = \left(\cos^2(1+t), \sin(2t)\right)$.

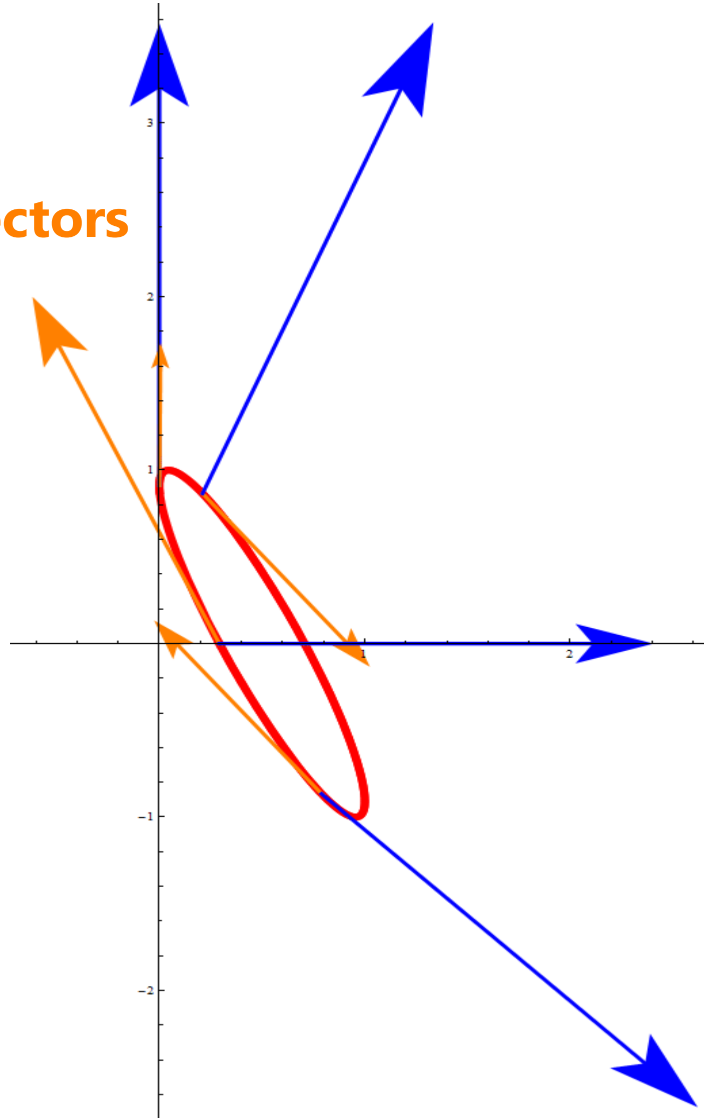
Plot $(x(t), y(t), f(x(t), y(t)))$, a path on the surface. Now, add in the gradient vectors on the path:



The surface is a like a mountain and the path is a hiking trail on that mountain, but the gradient vectors are basically a magic compass telling you the steepest direction from where you are standing. You can use this to figure out how steep the road ahead is.

Example 6: Tangent Vectors and Gradient Vectors on Our Path

Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and let $(x(t), y(t)) = \left(\cos^2(1+t), \sin(2t)\right)$. The gradient vectors along the path are blue and tangent vectors are orange.



Let's go point by point and describe what's happening on the hiking trail:

Example 6: Tangent Vectors and Gradient Vectors on Our Path

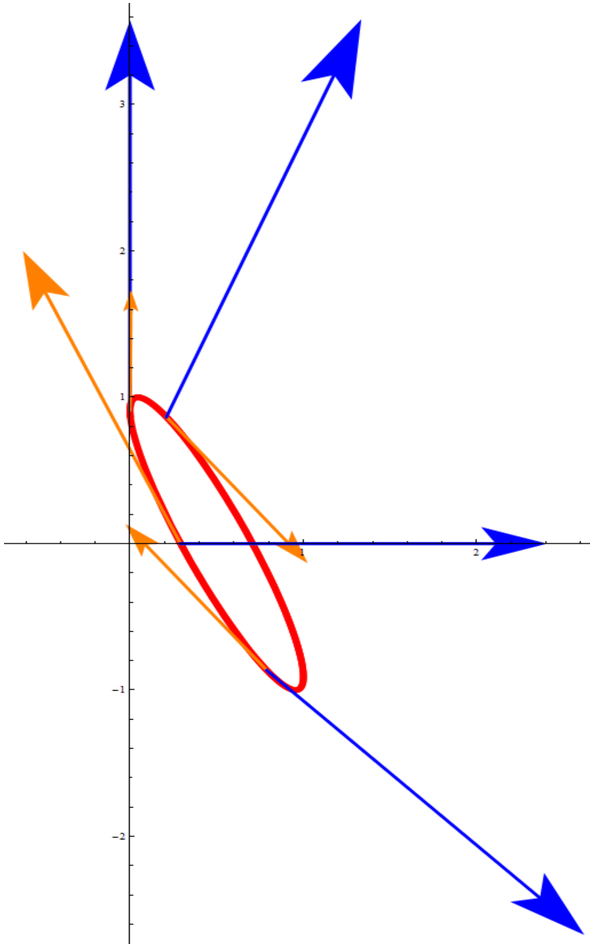
Let $f(x,y) = -(\cos^2(x) + \cos^2(y))^2$ and let $(x(t), y(t)) = \left(\cos^2(1+t), \sin(2t)\right)$.

The **gradient vectors** along the path are blue and **tangent vectors** are orange.

We saw the angle between the gradient, $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, and the tangent vector, $(x'(t), y'(t))$, is the key here! Recall that $V \bullet W = |V||W|\cos(\theta)$:

If the angle between $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $(x'(t), y'(t))$ is acute, $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet (x'(t), y'(t)) > 0$ and we are walking uphill.

If the angle between $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $(x'(t), y'(t))$ is obtuse, $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet (x'(t), y'(t)) < 0$ and we are walking downhill.



The Derivative of f(x(t),y(t)) With Respect to t

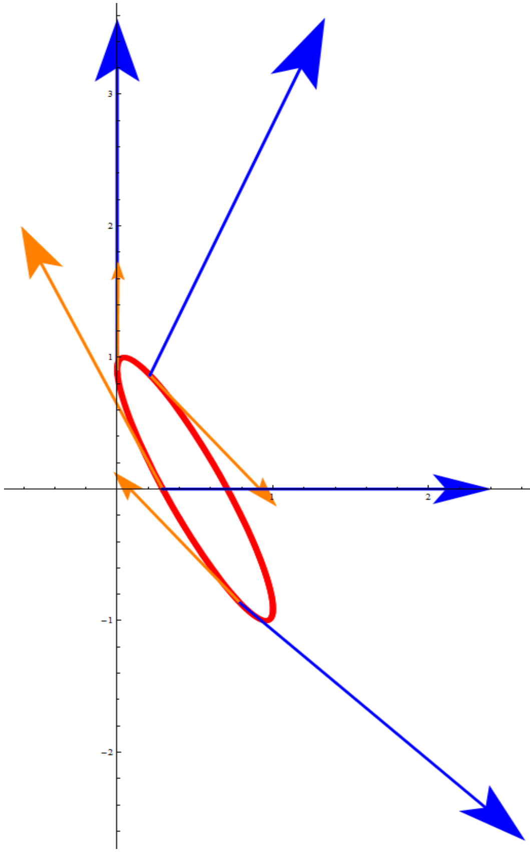
If the angle between $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ and (x'(t),y'(t)) is acute, $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\bullet(x'(t),y'(t))>0$ and we are walking uphill.

If the angle between $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ and (x'(t),y'(t)) is obtuse, $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\bullet(x'(t),y'(t))<0$ and we are walking downhill.

Walking uphill is like saying f(x(t),y(t)) is increasing at t.
Walking downhill is like saying f(x(t),y(t)) is decreasing at t.

The 2D Chain Rule:

$$\frac{df(x(t),y(t))}{dt}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\bullet(x'(t),y'(t))$$



The Chain Rule in n-Dimensions

The 2D Chain Rule: $$\frac{df(x(t),y(t))}{dt} = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\bullet(x'(t),y'(t))$$

Chain Rules in Other Dimensions:

1D : $$\frac{df(x(t))}{dt} = f'(x(t))x'(t)$$

3D : $$\frac{df(x(t),y(t),z(t))}{dt} = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\bullet(x'(t),y'(t),z'(t))$$

n-D: $$\frac{df(x_1(t),\ldots,x_n(t))}{dt} = \left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)\bullet(x_1'(t),\ldots,x_n'(t))$$

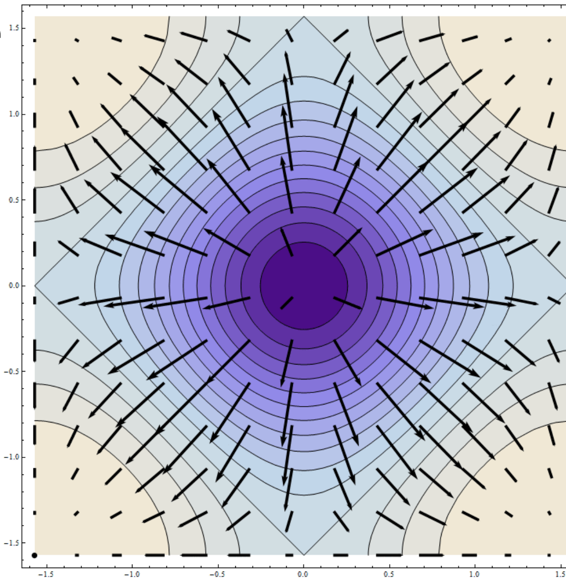
"The derivative of the outside times the derivative of the inside."

(Gradient) **(Dot Product)** **(Tangent Vector)**

Chain Rule Proves the Gradient is Perpendicular to the Level Curve

Theorem: The gradient is always perpendicular to the level curve through its tail.

Proof: We will only show this for a surface $z = f(x,y)$ whose level curve $c = f(x,y)$ can be parameterized by $(x(t), y(t))$. Then a tangent vector on the level curve can be described by $(x'(t), y'(t))$.



Next, the gradient is $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

For them to be perpendicular, we want their dot product to be 0:

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet (x'(t), y'(t)) = \frac{df(x(t), y(t))}{dt}$$

By Chain Rule...

$$= 0$$

But $c = f(x,y)$, and the derivative of a constant is 0...

So the gradient is $\perp$ to the level curve.

Another way to think of this is a level curve is defined as a path on the surface where your height stays constant. So $\frac{df(x(t), y(t))}{dt} = 0$...

Example 7: Using the Chain Rule

Let $f(x,y) = x^2 + xy$ and let $(x(t), y(t)) = \left(\cos(t), \sin(t)\right)$. Find $\dfrac{df}{dt}$:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2x + y, x\right) = \left(2\cos(t) + \sin(t), \cos(t)\right)$$

$$(x'(t), y'(t)) = (-\sin(t), \cos(t))$$

$$\frac{df(x(t), y(t))}{dt} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet (x'(t), y'(t))$$

$$= \left(2\cos(t) + \sin(t), \cos(t)\right) \bullet (-\sin(t), \cos(t))$$

$$= -2\sin(t)\cos(t) - \sin^2(t) + \cos^2(t)$$

$$= \cos(2t) - \sin(2t)$$

Example 8: Another Visit to Our Surface

Let $f(x,y)=x^3+y^3-3x-3y$ and suppose you are standing on the surface at the point $(2,3,20)$. If you decide to walk in the direction of $(-7,2)$, do you go uphill or downhill on f(x,y) when you take your first step?

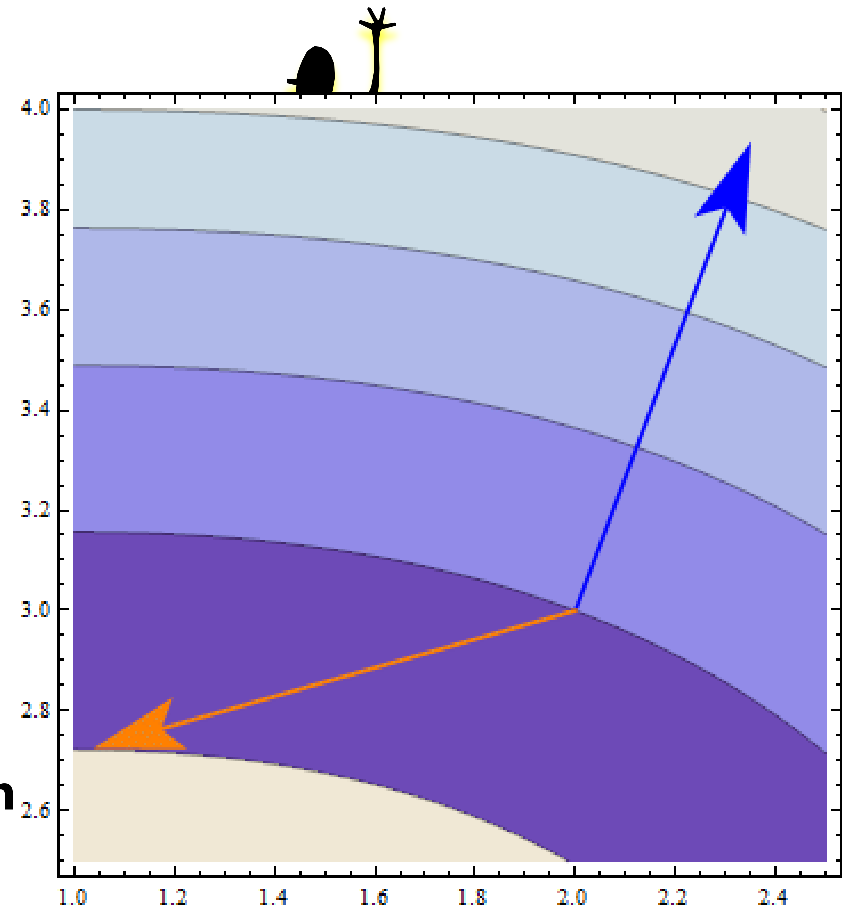
$$\nabla f(x,y)=\left(3x^2-3,3y^2-3\right)$$

$$\nabla f(2,3)=\left(9,24\right)$$

$$\left(9,24\right)\bullet\left(-7,2\right)=-15$$

Since this dot product is negative, we know our first step will be downhill.

This negative tells us the angle between the gradient at (2,3) and our direction vector is obtuse. We are walking against the direction advised by our Gradient Compass: downhill!

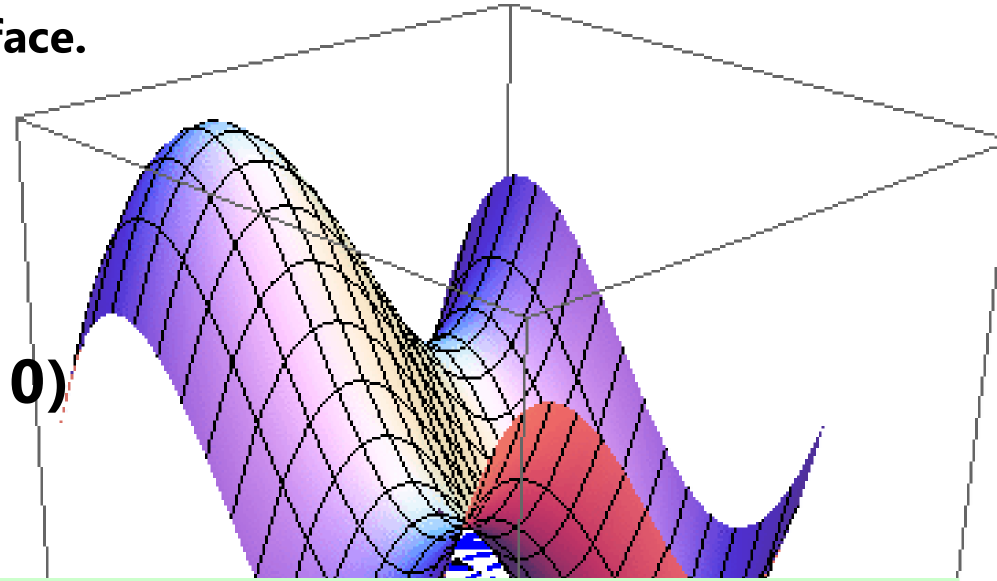


Example 9: Identifying Local Extrema from the Gradient

Let $f(x,y)=x^3+y^3-3x-3y$. We previously found $\nabla f(x,y)=\left(3x^2-3,3y^2-3\right)$.

Find all local maxima/minima on this surface.

Theorem: The candidates for local extrema are where $\nabla f=(0,0)$, or where it is undefined.



$$\nabla f(x,y)=\left(3x^2-3,3y^2-3\right)=(0,0)$$

$$x=\pm 1, y=\pm 1$$

Critical Points:

$$(1,1),(1,-1),$$
$$(-1,1),(-1,-1)$$

Why are the points where $\nabla f=(0,0)$ candidates for local extrema? Well, points where we have $|\nabla f|=|(0,0)|=0$ indicates that the surface is perfectly flat here. That is, the tangent planes to the surface at those points are horizontal. So these are locations where we could have a local maximum or minimum.

Example 9: Identifying Local Extrema from the Gradient

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f(x,y) = \left(3x^2 - 3, 3y^2 - 3\right)$.

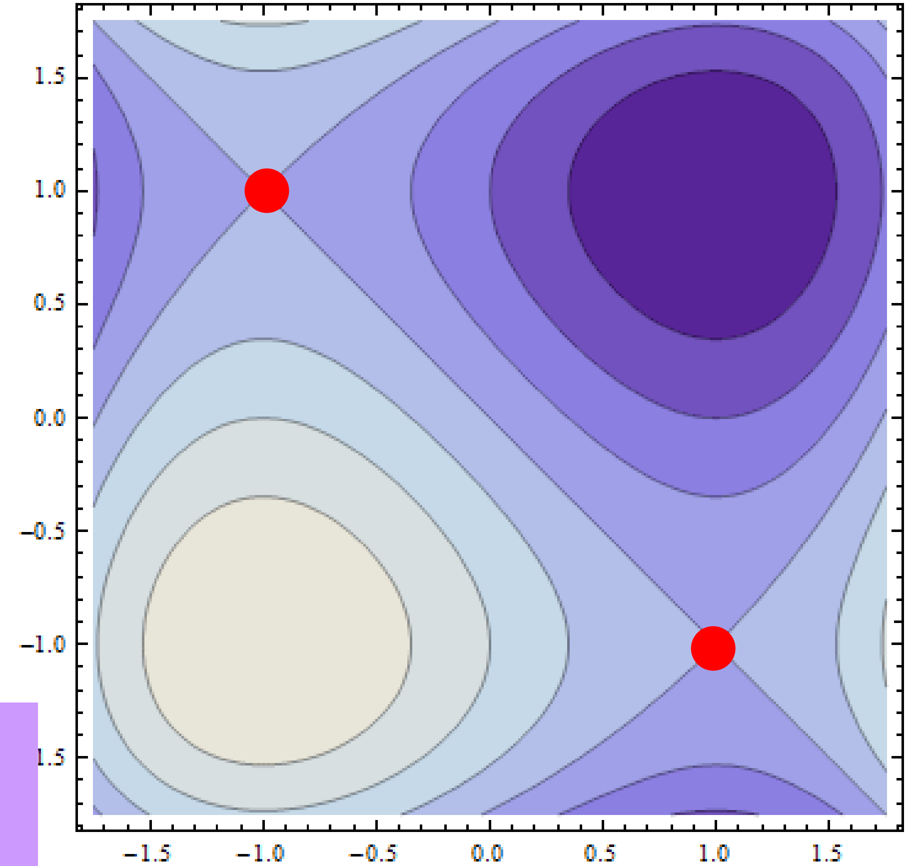
Find all local maxima/minima on this surface.

Critical Points:

$\left(1,1\right),\left(1,-1\right),\left(-1,1\right),\left(-1,-1\right)$

$(1,-1)$ and $(-1,1)$ are called saddle points:

They bring you up in one direction down in the other. These are not extrema.



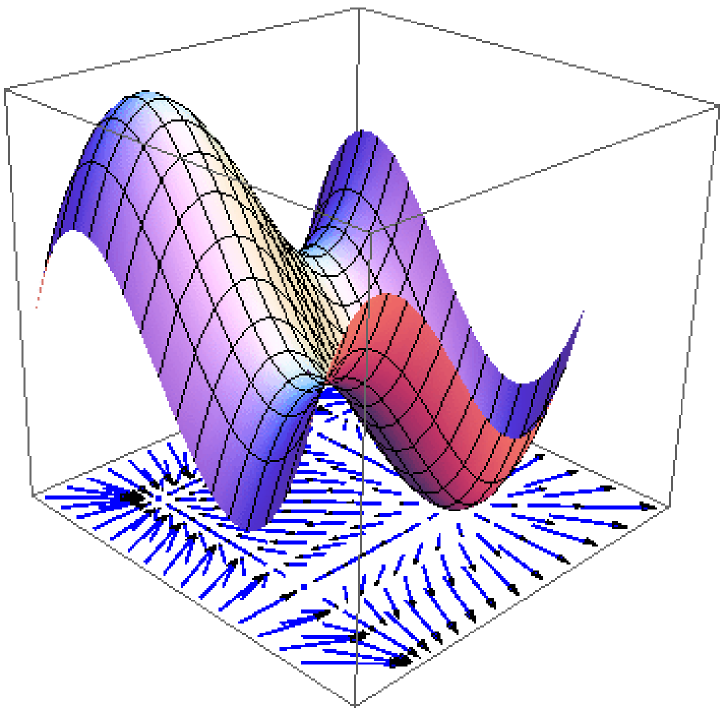
Task: Look at the surface in Mathematica and convince yourself these are not extrema.

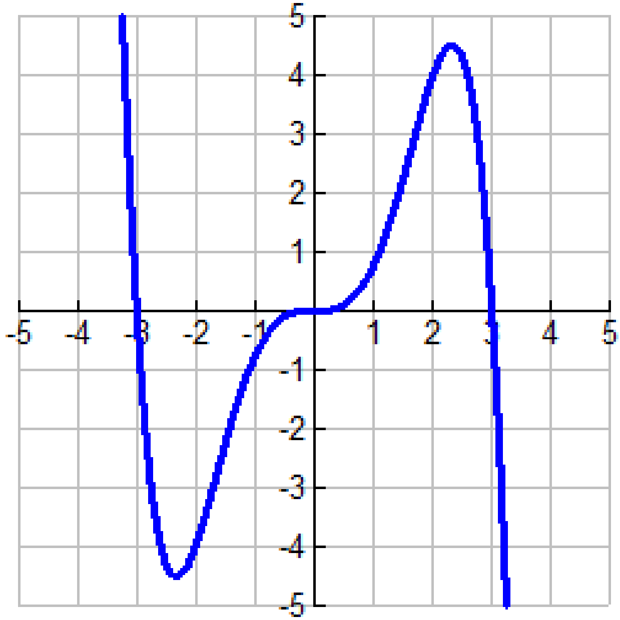
Example 9: Identifying Local Extrema from the Gradient

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f(x,y) = \left(3x^2 - 3, 3y^2 - 3\right)$.

Find all local maxima/minima on this surface.

<u>Saddle point analogy:</u>





A <u>Saddle Point</u> is to 3D as a Non-Max/Min Horizontal Tangent is to 2D.

Example 9: Identifying Local Extrema from the Gradient

Let $f(x,y)=x^3+y^3-3x-3y$. We previously found $\nabla f(x,y)=\left(3x^2-3,3y^2-3\right)$.

Find all local maxima/minima on this surface.

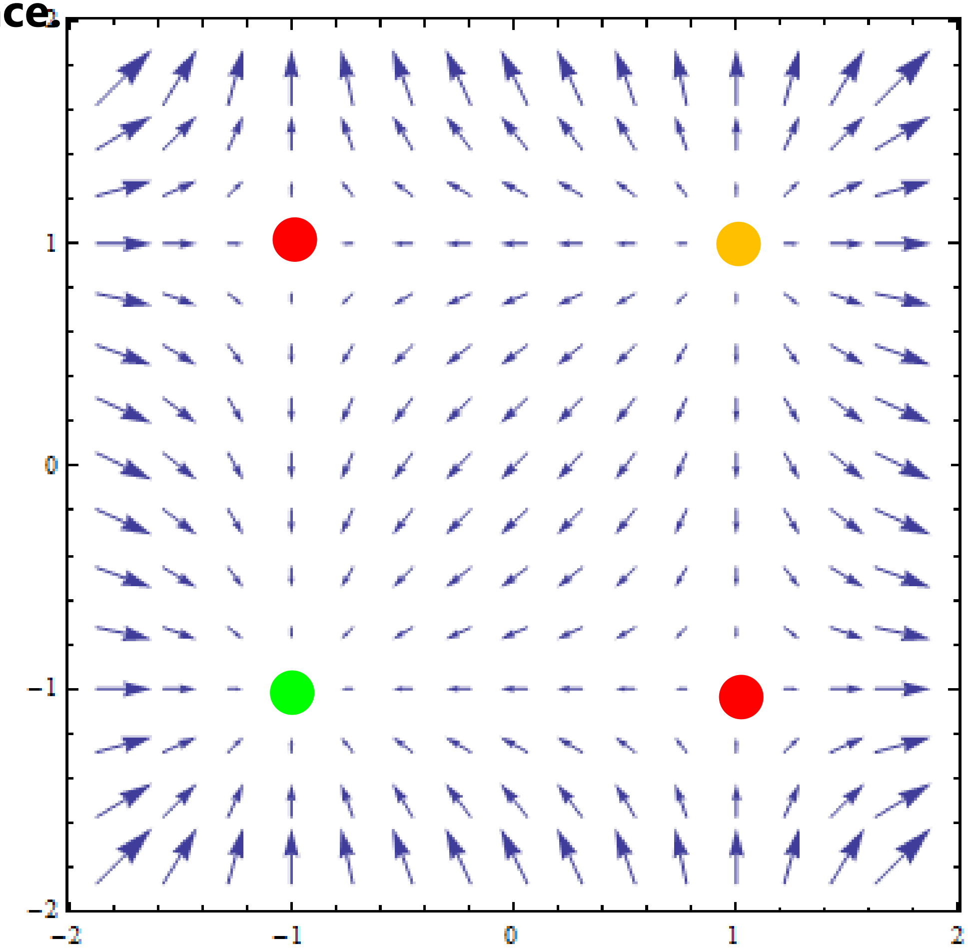
Remaining Candidates:

$(1,1)$ and $(-1,-1)$

The gradient vectors show that $(-1,-1)$ is a local max (the surrounding vectors all point to $(-1,-1)$).

The gradient vectors show that $(1,1)$ is a local min (the surrounding vectors all point away from $(1,1)$).

Saddle points have gradient vectors both flowing in and out.



Example 9: Identifying Local Extrema from the Gradient

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f(x,y) = \left(3x^2 - 3, 3y^2 - 3\right)$.

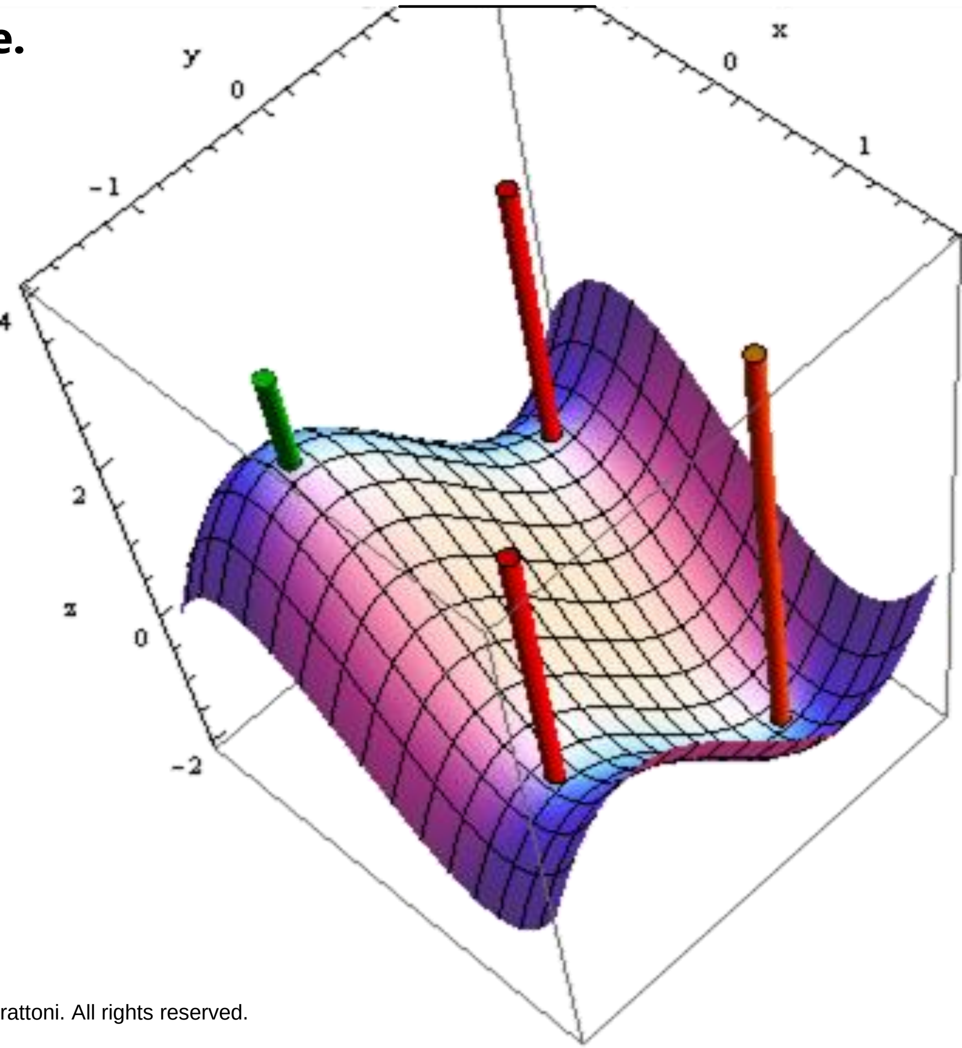
Find all local maxima/minima on this surface.

Summary:

Local Maximum: $(-1,-1)$

Local Minimum: $(1,1)$

Saddle Points: $(1,-1)$ and $(-1,1)$



Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

You might wonder how to identify maxima, minima, and saddle points without having to use Mathematica to plot the gradient field. For this, we generalize the second derivative test.

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Recall: The second derivative states that if $f'(a) = 0$, then:

If $f''(a) < 0$, then we have a local maximum at $x = a$

If $f''(a) > 0$, then we have a local minimum at $x = a$

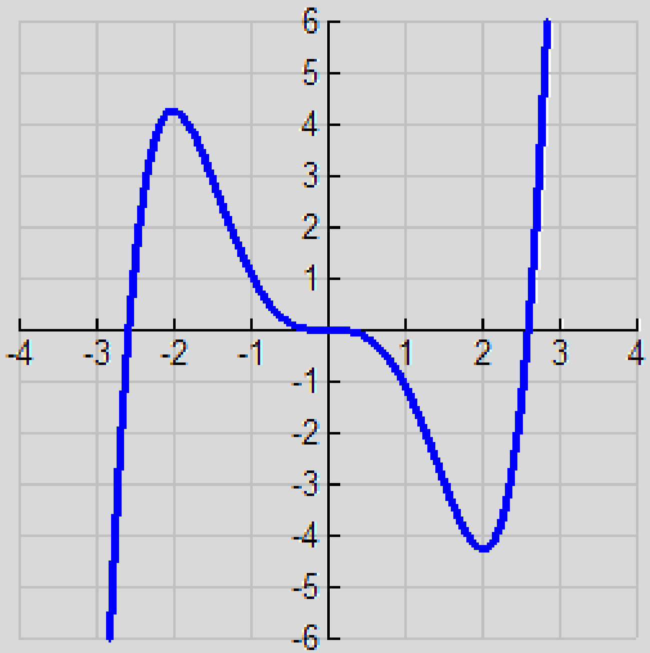
If $f''(a) = 0$, then the test is inconclusive.

Example: Let $f(x) = \frac{x^5}{5} - \frac{4x^3}{3}$:

$f'(-2) = 0$ & $f''(-2) < 0$: local maximum at $x = -2$.

$f'(2) = 0$ & $f''(2) > 0$: local minimum at $x = 2$.

$f'(0) = 0$ & $f''(0) = 0$: test inconclusive at $x = 0$.



Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Now for a maximum, we need to be concave down in the x-direction AND y-direction. Likewise, for a minimum, we need to be concave up in the x-direction AND y-direction. If we mix between the two, we have a saddle point.

The tool that lets us analyze this is the Hessian Matrix:

$$Hf(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Hessian Determinant: $D(x,y) = \left|Hf(x,y)\right| = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}$

$D(a,b)$	$f_{xx}(a,b)$	Conclusion
$+$	$-$	$(a,b,f(a,b))$ is a local maximum.
$+$	$+$	$(a,b,f(a,b))$ is a local minimum.
$-$	n / a	$(a,b,f(a,b))$ is a saddle point.
0	n / a	The test is inconclusive.

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

D(a,b)	f_{xx}(a,b)	Conclusion
+	–	(a,b,f(a,b)) is a local maximum.
+	+	(a,b,f(a,b)) is a local minimum.
–	n / a	(a,b,f(a,b)) is a saddle point.
0	n / a	The test is inconclusive.

How does this work? Well, if D(a,b) is positive, it is telling us that the "second derivatives in the x-direction and y-direction" are pointing the same way. So if D(a,b) is positive and f_{xx}(a,b) is negative, then we are "concave down" in both directions which is a local maximum. If D(a,b) is positive and f_{xx}(a,b) is positive, then we are "concave up" in both directions, which is a local minimum.

If D(a,b) is negative, then the x-direction and y-direction second derivatives are "pointing in different directions," or a saddle point.

If D(a,b) is zero, we can't tell.

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f = \left(3x^2 - 3, 3y^2 - 3\right)$ with critical points at $(-1,-1)$, $(1,1)$, $(1,-1)$, and $(-1,1)$. Use the Hessian determinant to identify maxima, minima, and saddle points:

$$D(x,y) = \left|Hf(x,y)\right| = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(x^3 + y^3 - 3x - 3y\right)\right) & \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y}\left(x^3 + y^3 - 3x - 3y\right)\right) \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x}\left(x^3 + y^3 - 3x - 3y\right)\right) & \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\left(x^3 + y^3 - 3x - 3y\right)\right) \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial x}\left(3x^2 - 3\right) & \dfrac{\partial}{\partial x}\left(3y^2 - 3\right) \\ \dfrac{\partial}{\partial y}\left(3x^2 - 3\right) & \dfrac{\partial}{\partial y}\left(3y^2 - 3\right) \end{vmatrix}$$

$$= \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix}$$

$$= 36xy$$

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f = \left(3x^2 - 3, 3y^2 - 3\right)$ with critical points at $(-1,-1)$, $(1,1)$, $(1,-1)$, and $(-1,1)$. Use the Hessian determinant to identify maxima, minima, and saddle points:

$$D(x,y) = \left|Hf(x,y)\right| = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$$

Test (1,1):

$D(1,1) = 36 > 0$

$f_{xx}(1,1) = 6x\big|_{(1,1)} = 6 > 0$

$D(a,b)$	$f_{xx}(a,b)$	Conclusion
+	−	$(a,b,f(a,b))$ is a local maximum.
+	+	$(a,b,f(a,b))$ is a local minimum.
−	n / a	$(a,b,f(a,b))$ is a saddle point.
0	n / a	The test is inconclusive.

Hence, $(1,1,f(1,1))$ is a local minimum!

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y)=x^3+y^3-3x-3y$. We previously found $\nabla f=\left(3x^2-3,3y^2-3\right)$ with critical points at $(-1,-1)$, $(1,1)$, $(1,-1)$, and $(-1,1)$. Use the Hessian determinant to identify maxima, minima, and saddle points:

$$D(x,y)=\left|Hf(x,y)\right|=\begin{vmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2}\end{vmatrix}=\begin{vmatrix}6x & 0\\ 0 & 6y\end{vmatrix}=36xy$$

Test $(-1,-1)$:

$D(-1,-1)=36>0$

$f_{xx}(-1,-1)=6x\big|_{(-1,-1)}=-6<0$

$D(a,b)$	$f_{xx}(a,b)$	Conclusion
+	−	$(a,b,f(a,b))$ is a local maximum.
+	+	$(a,b,f(a,b))$ is a local minimum.
−	n / a	$(a,b,f(a,b))$ is a saddle point.
0	n / a	The test is inconclusive.

Hence, $(-1,-1,f(-1,-1))$ is a local maximum!

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y)=x^3+y^3-3x-3y$. We previously found $\nabla f=\left(3x^2-3,3y^2-3\right)$ with critical points at $(-1,-1)$, $(1,1)$, $(1,-1)$, and $(-1,1)$. Use the Hessian determinant to identify maxima, minima, and saddle points:

$$D(x,y)=\left|Hf(x,y)\right|=\begin{vmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix}=\begin{vmatrix}6x & 0\\ 0 & 6y\end{vmatrix}=36xy$$

Test $(1,-1)$:

$$D(1,-1)=-36<0$$

$D(a,b)$	$f_{xx}(a,b)$	Conclusion
+	−	$(a,b,f(a,b))$ is a local maximum.
+	+	$(a,b,f(a,b))$ is a local minimum.
−	n / a	$(a,b,f(a,b))$ is a saddle point.
0	n / a	The test is inconclusive.

Hence, $(1,-1,f(1,-1))$ is a saddle point!

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f = \left(3x^2 - 3, 3y^2 - 3\right)$ with critical points at $(-1,-1)$, $(1,1)$, $(1,-1)$, and $(-1,1)$. Use the Hessian determinant to identify maxima, minima, and saddle points:

$$D(x,y) = \left|Hf(x,y)\right| = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$$

Test $(-1,1)$:

$$D(-1,1) = -36 < 0$$

$D(a,b)$	$f_{xx}(a,b)$	Conclusion
+	−	$(a,b,f(a,b))$ is a local maximum.
+	+	$(a,b,f(a,b))$ is a local minimum.
−	n / a	$(a,b,f(a,b))$ is a saddle point.
0	n / a	The test is inconclusive.

Hence, $(-1,1,f(-1,1))$ is a saddle point!

Example 10: The Second Partial Derivative Test to Identify Maximums and Minimums

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f = \left(3x^2 - 3, 3y^2 - 3\right)$.

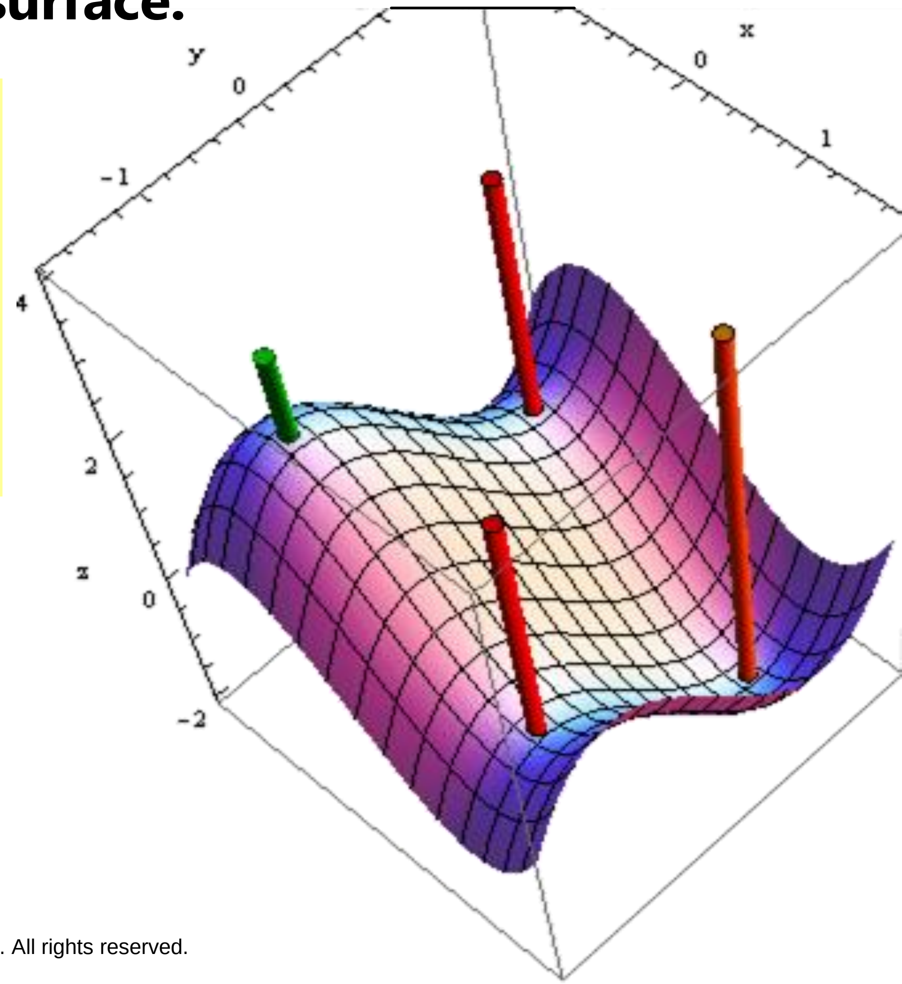
Find all local maxima/minima on this surface.

Summary:

Local Maximum: $(-1,-1)$

Local Minimum: $(1,1)$

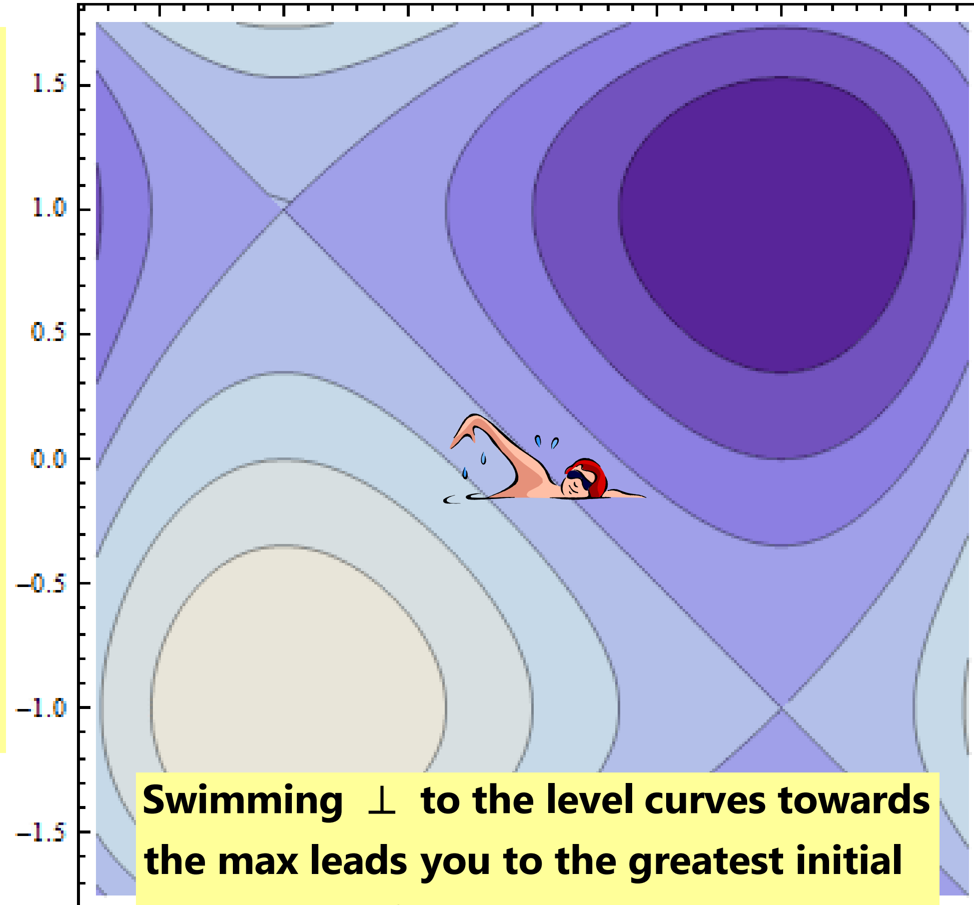
Saddle Points: $(1,-1)$ and $(-1,1)$



Example 11: Level Surfaces and Entering the Fourth Dimension

Let $f(x,y) = x^3 + y^3 - 3x - 3y$. We previously found $\nabla f = \left(3x^2 - 3, 3y^2 - 3\right)$ with a local maximum at $(-1,-1)$ and a local maximum at $(1,1)$.

One idea: if your friend likes to swim, you could ask him/her to swim on the contour plot. The local max would be where the water is hottest and the minimum would be the coldest. You've helped your friend experience the third dimension as temperature rather than a spatial dimension.



Swimming $\perp$ to the level curves towards the max leads you to the greatest initial temperature increase.

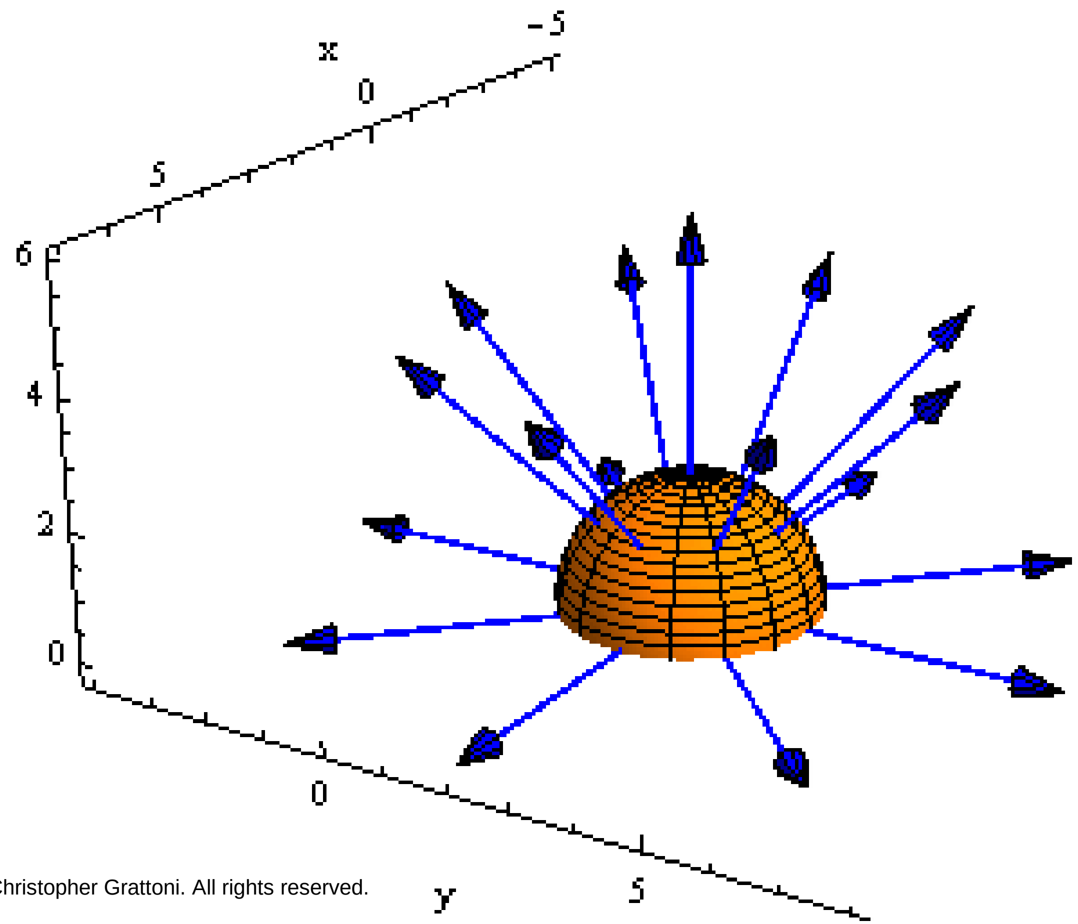
Example 11: Level Surfaces and Entering the Fourth Dimension

Let $f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$. We can plot level surfaces of this 4D-surface by letting $k = (x-1)^2 + (y-2)^2 + z^2$ for various values of k.

Task: Look at the level surfaces in Mathematica by clicking the slider (don't drag it though)

We could experience 4D like the swimmer experienced 3D. We could swim in 3D space and experience the fourth dimension as temperature.

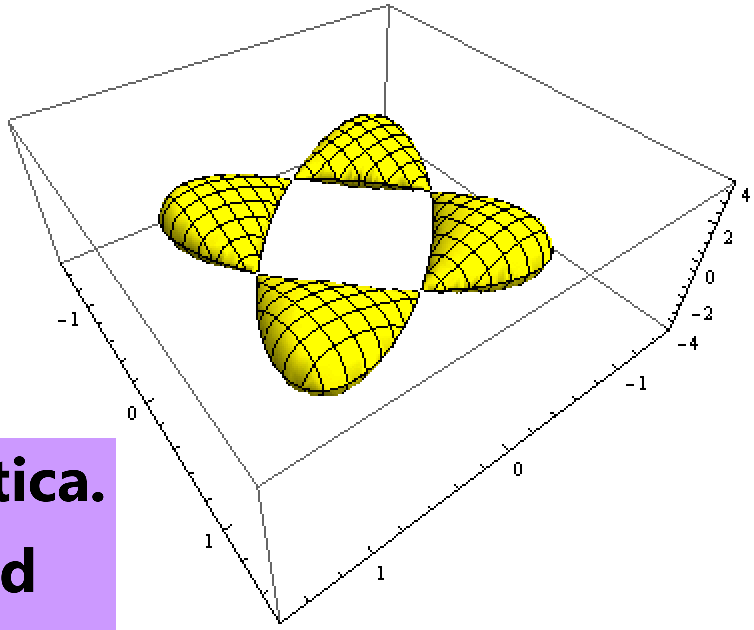
The gradient vectors tell us the path to the greatest initial temperature increase from a given point in space.



Example 11: A Different Analogy for the Fourth Dimension?

Let $T(x,y,z) = 8(x^4 - x^2 + y^4 - y^2) + z + 2$. We can plot level surfaces of this 4D-surface by letting $k = 8(x^4 - x^2 + y^4 - y^2) + z + 2$ for various values of k.

This time, treat T(x,y,z) as a function that takes a position in space (x,y,z) and outputs the temperature at that point. So our swimmer can experience an extra dimension as the temperature at that particular point. Level surfaces can be found by graphing various $T(x,y,z) = k$.



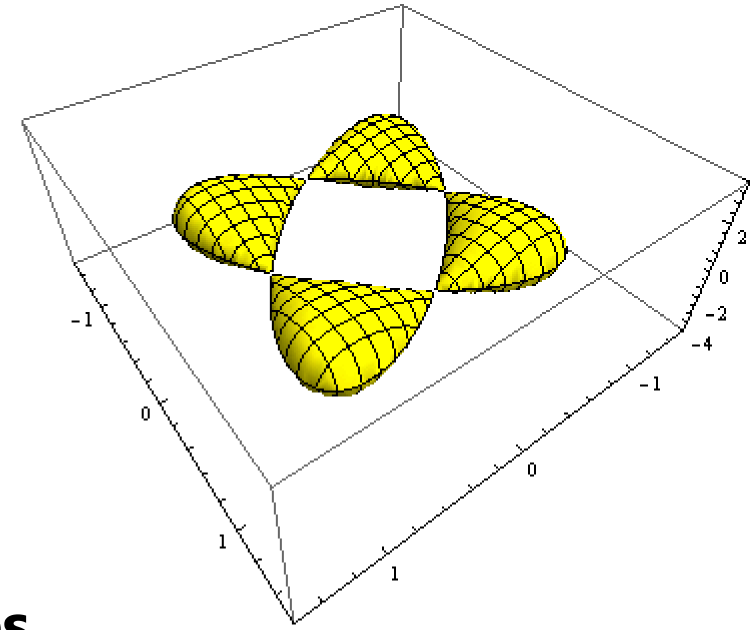
Task: Look at this surface in Mathematica. Click (don't drag) around the slider and think about how a 3D swimmer would experience this diving tank.

Example 12: A Different Analogy for the Fourth Dimension?

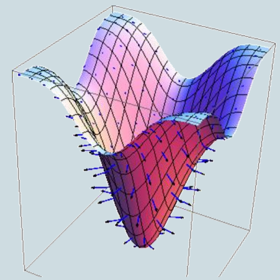
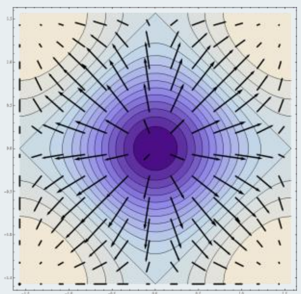
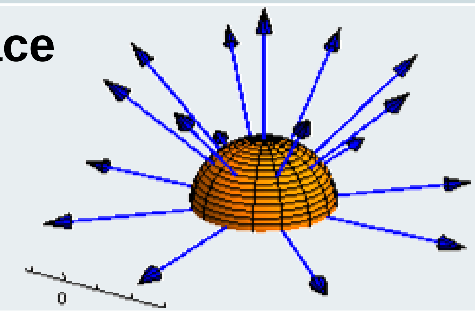
Let $T(x,y,z) = 8(x^4 - x^2 + y^4 - y^2) + z + 2$. We can plot level surfaces of this 4D-surface by letting $k = 8(x^4 - x^2 + y^4 - y^2) + z + 2$ for various values of k.

Discussion Questions:

1) Where should the swimmer go for the coolest locations?
2) The hottest?
3) What does a level surface mean in the context of this scenario?
4) Given the particular level surface the swimmer is on, in what direction does he swim for the greatest initial temperature increase?



Final Thoughts: f(x,y) versus f(x,y,z)

	z = f(x,y)	w = f(x,y,z)
Graph	**Surface in 3-D** 	**(Hyper)Surface in 4-D** *Can't truly graph it!*
Level Sets	**Level Curve** *k = f(x,y)* 	**Level Surface** *k = f(x,y,z)* 
Gradient Vectors	**2D Vectors** $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ **(2D vectors that are $\perp$ to the level curves, NOT the 3D surface itself)**	**3D Vectors** $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ **(3D vectors that are $\perp$ to the level surface, NOT the 4D surface itself)**

Up Next

- Read about LaGrange multipliers!