Lesson 4:

Lagrange Multipliers (Constrained Optimization)

Example 1: Lagrange's Method

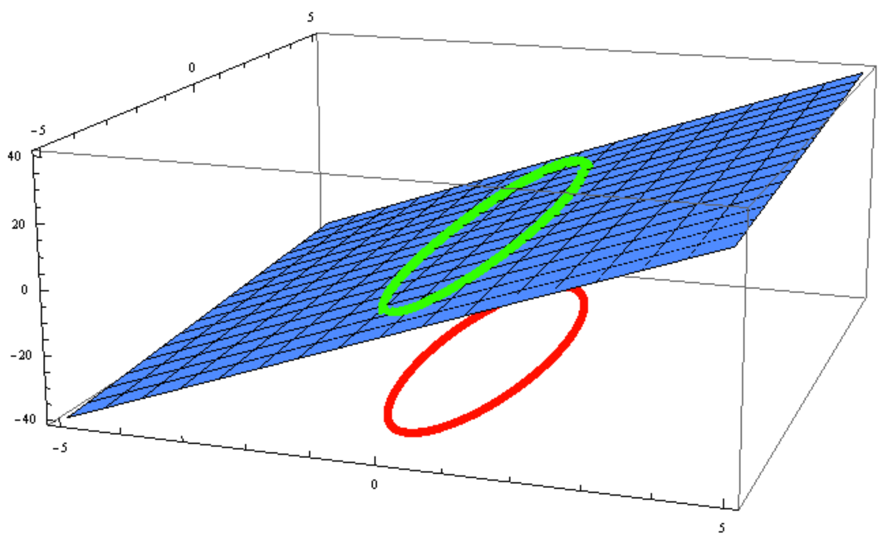
Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$.

1) $f(x,y) = 7x + y$ is a surface

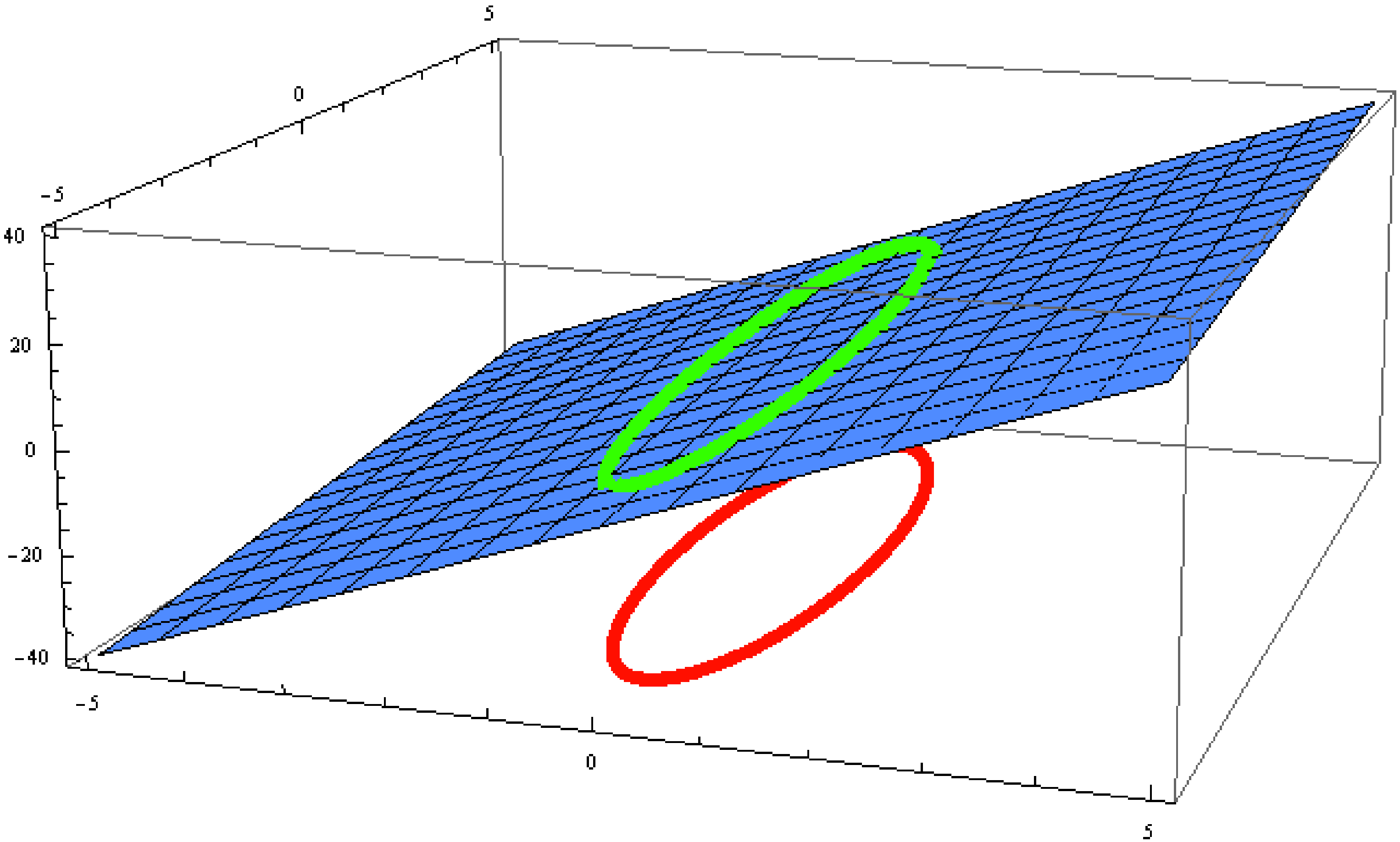
2) $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$ is a path that we can map onto the surface.

3) This question wants us to find the highest point on the green ellipse.

That is, we want (x,y,z) on the green ellipse with z as large as possible.



Example 1: Lagrange's Method



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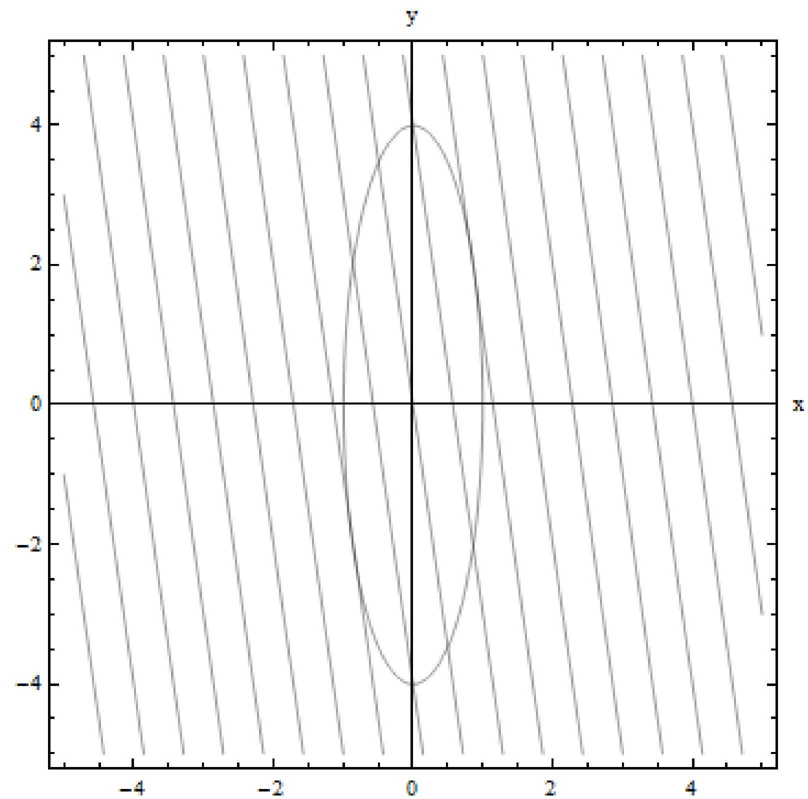
One way to try to find the highest point on the green ellipse would be to simplify the picture we are looking at. Let's reduce the 3D surface, f(x,y), to numerous level curves:



Where does it look like the points with highest/lowest z-coordinate on the elliptical path on the surface occur?

Indeed! The highest and lowest points will be where the level curves are tangent to the ellipse. (This is worth pondering if you are wondering why!)

Essentially, we are looking for which level curve(s) of f(x,y) are tangent to $g(x, y) = 1$.

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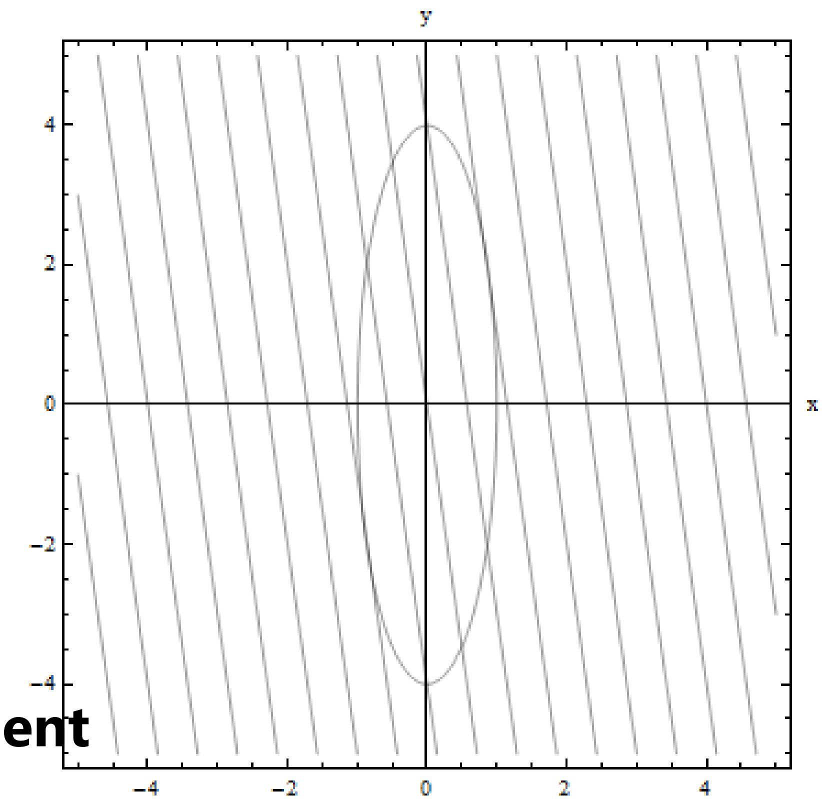
Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$.

The level curves are tangent when the gradient vectors to $z = f(x,y)$ and $z = g(x,y)$ are pointing in the same direction. That is, when $\nabla f(x,y)$ and $\nabla g(x,y)$ are multiples of each other.

So we are looking for the points (x,y) satisfying two conditions:

1) $\nabla f(x,y) = s\nabla g(x,y)$
2) $g(x,y) = 1$

In a sentence, we are looking for the points (x,y) on the curve $g(x,y) = 1$ where the gradient of f and the gradient of g are pointing in the same direction.



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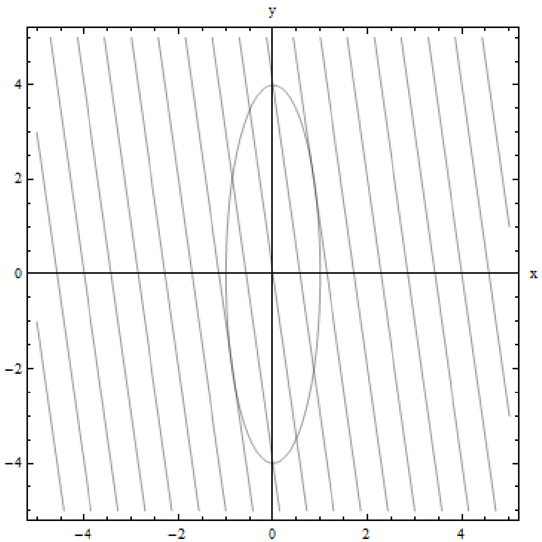
Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$.

1) $\nabla f(x,y) = s\nabla g(x,y)$

$$(7,1) = s\left(2x, \frac{y}{8}\right)$$

$$2xs = 7 \quad \text{and} \quad \frac{sy}{8} = 1$$

$$x = \frac{7}{2s} \quad \text{and} \quad y = \frac{8}{s}$$



2) $g(x,y) = 1$

$$x^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\left(\frac{7}{2s}\right)^2 + \left(\frac{\left(\frac{8}{s}\right)}{4}\right)^2 = 1$$

$$\frac{65}{4s^2} = 1$$

$$s = \pm\frac{\sqrt{65}}{2}$$

3) Plug $s = \frac{\sqrt{65}}{2}$ into

$x = \frac{7}{2s}$ and $y = \frac{8}{s}$:

$x = 0.868, y = 1.985$

$(0.868, 1.985)$

Plug $s = -\frac{\sqrt{65}}{2}$ into

$x = \frac{7}{2s}$ and $y = \frac{8}{s}$:

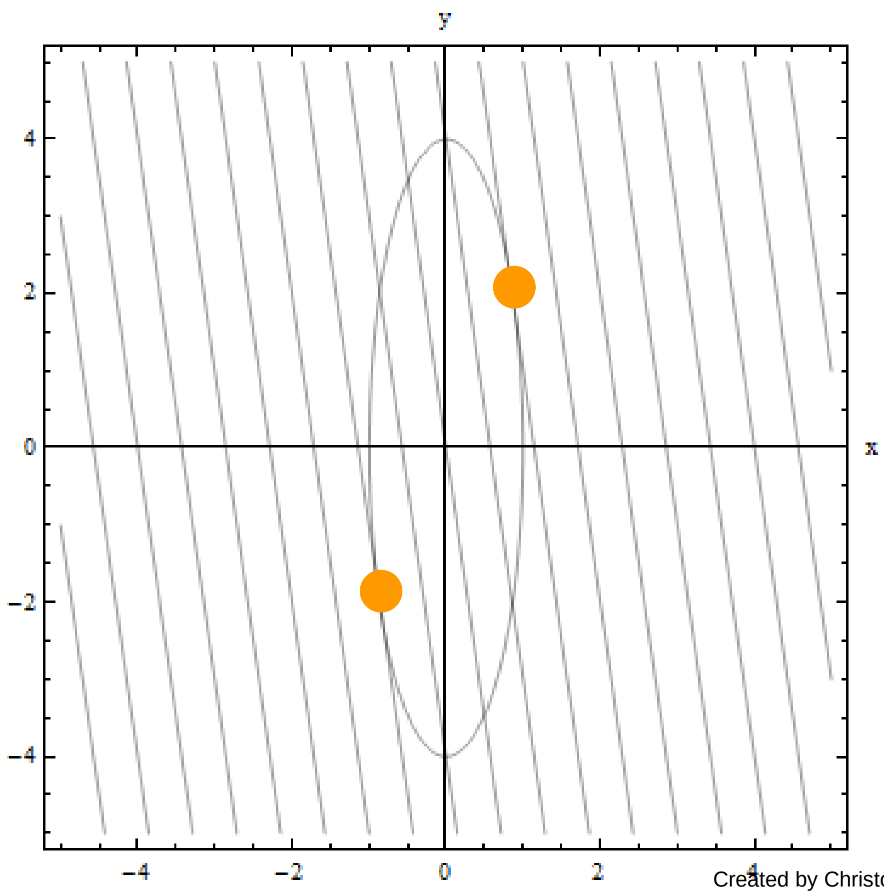
$x = -0.868, y = -1.985$

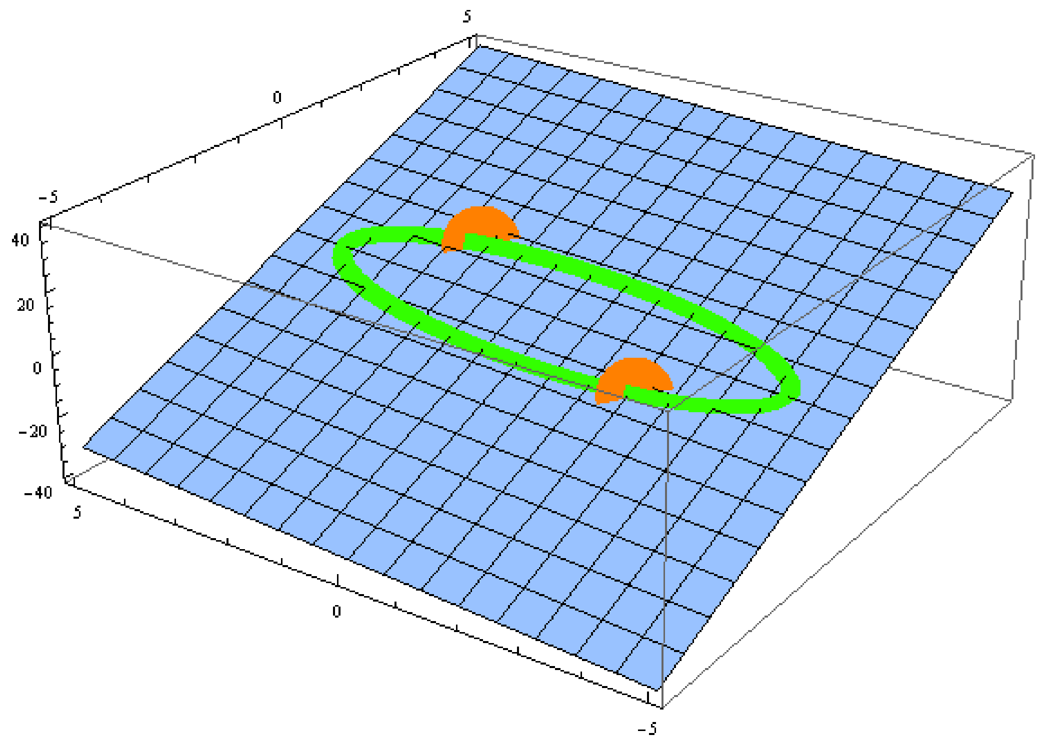
$(-0.868, -1.985)$

Example 1: Lagrange's Method

Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$.

Our CANDIDATES are $(0.868, 1.985)$ and $(-0.868, -1.985)$:





Example 1: Lagrange's Method

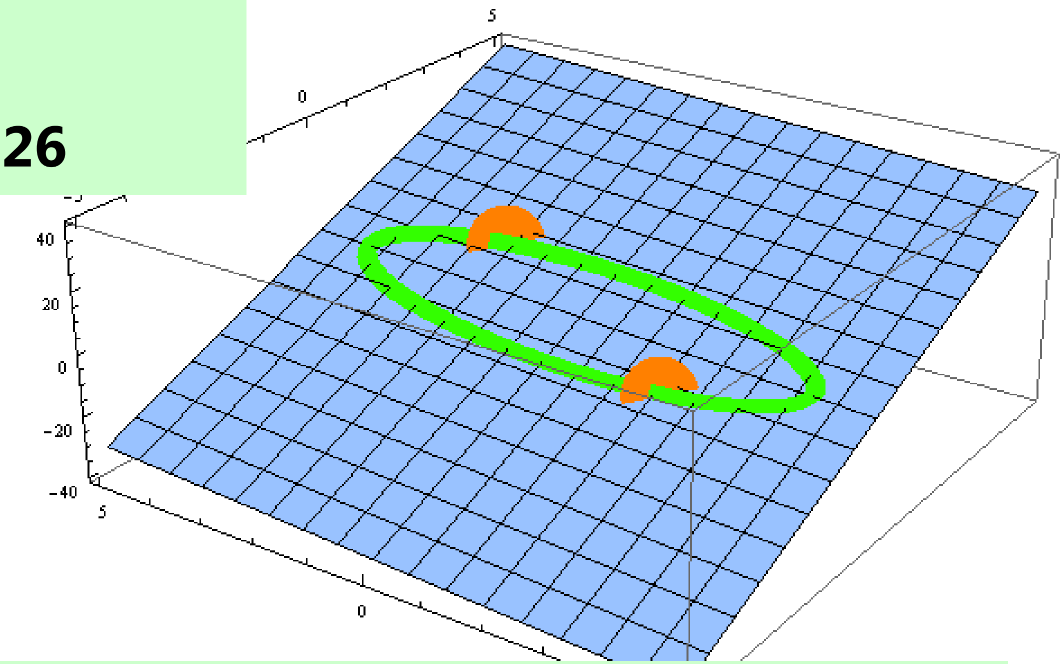
Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1.$

Our CANDIDATES are $(0.868, 1.985)$ and $(-0.868, -1.985)$:

Test candidates to find the maximum:

$f(0.868, 1.985) \approx 8.06226$

$f(-0.868, -1.985) \approx -8.06226$



Maximum: $(0.868, 1.985, 8.062)$

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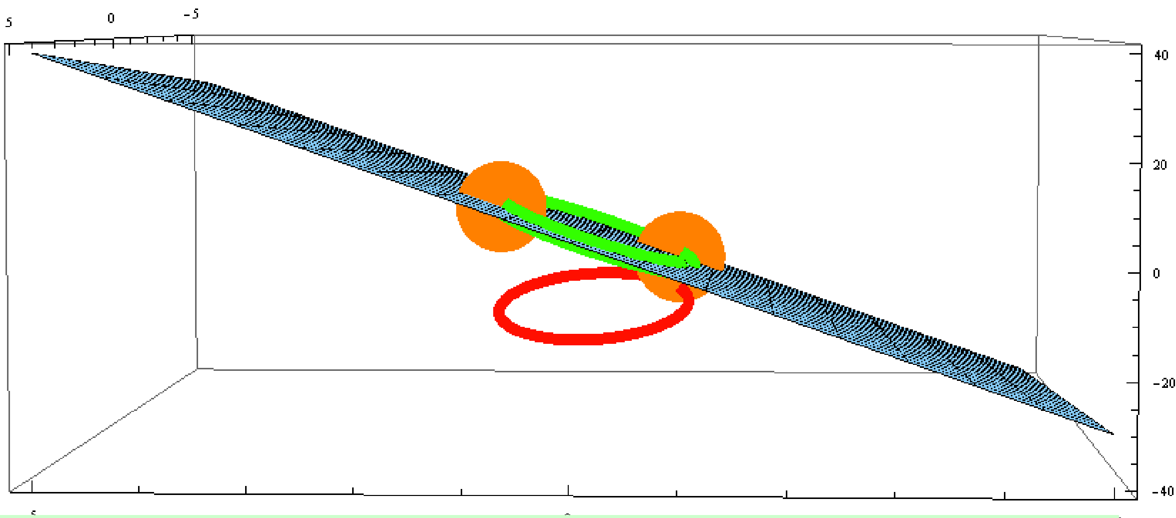
Maximize $f(x,y) = 7x + y$ given the constraint $g(x,y) = x^2 + \left(\frac{y}{4}\right)^2 = 1$.

Our CANDIDATES are $(0.868, 1.985)$ and $(-0.868, -1.985)$:

In the CDF demo, try rotating this plot until you convince yourself that we found a logical answer!



Maximum: $(0.868, 1.985, 8.062)$

Summary: Lagrange's Method

Maximize or minimize $z = f(x, y)$ given the constraint $g(x, y) = c$.

1) **Set up a system of equations as follows**

   i. $\nabla f(x, y) = s\nabla g(x, y)$

   ii. $g(x, y) = c$

2) **Solve the system for s and plug back in to find x and y.**

3) **Test all candidates (x_0, y_0) in f to determine where your maximum (or minimum) is.**

Note: This process does not need to be limited to a 3D-surface with a 2D-curve that acts as a constraint!

Next Steps

1) **Read more in the text**
2) **Here are some more resources:**

http://tutorial.math.lamar.edu/Classes/CalcIII/LagrangeMultipliers.aspx

http://www2.bc.cc.ca.us/resperic/mathb6c/Three%20example%20Lagrange%20multiplier%20problems.pdf

http://www.math.uakron.edu/~norfolk/bestbox223.pdf