Lesson 5

Double Integrals, Volume Calculations, and the Gauss-Green Formula

Example 1: A Double Integral Over a Rectangular Region

Let $f(x,y) = \sin(y) + \cos(x) + 2$ and let R be the region in the xy-plane bounded by $x = 0, x = \pi, y = -2\pi,$ and $y = \pi$. Calculate $\iint_R f(x,y)\,dA$:

$$\int_0^{\pi}\int_{-2\pi}^{\pi} \left(\sin(y) + \cos(x) + 2\right) dy\,dx$$

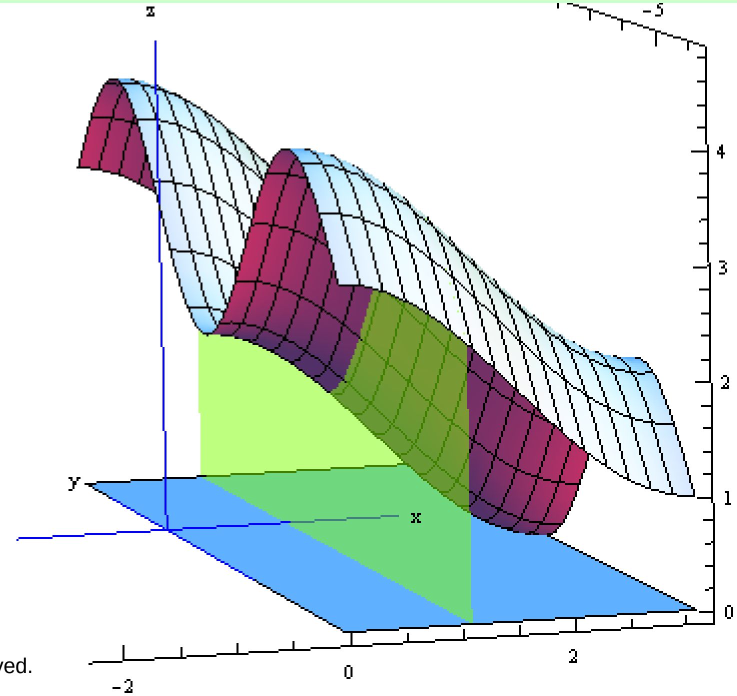
$$= \int_0^{\pi}\left[-\cos(y) + y\cos(x) + 2y\right]_{y=-2\pi}^{y=\pi} dx$$

$$= \int_0^{\pi} 2 + 6\pi + 3\pi\cos(x)\,dx$$

$$= \left[2x + 6\pi x + 3\pi\sin(x)\right]_{x=0}^{x=\pi}$$

$$= 2\pi + 6\pi^2$$

$2 + 6\pi + 3\pi\cos(x)$ is a function that sweeps out vertical area cross sections from $x = 0$ to $x = \pi$. If we integrate these area cross sections, we accumulate the volume of the solid:



Example 2: Changing the Order of Integration

Calculate $\iint_R f(x,y)\, dA$ by computing $\iint_R f(x,y)\, dx\, dy$ instead of $\iint_R f(x,y)\, dy\, dx$.

$$\int_{-2\pi}^{\pi}\int_{0}^{\pi}\left(\sin(y)+\cos(x)+2\right)dx\, dy$$

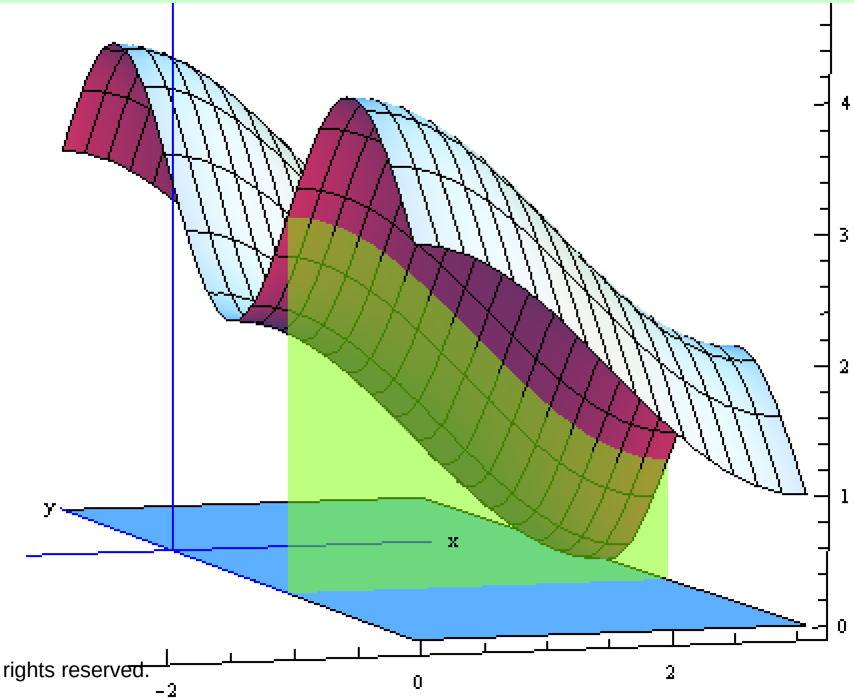
$$= \int_{-2\pi}^{\pi}\left[x\sin(y)+\sin(x)+2x\right]_{x=0}^{x=\pi} dy$$

$$= \int_{-2\pi}^{\pi} 2\pi+\pi\sin(y)\, dy$$

$$= \left[2\pi y-\pi\cos(y)\right]_{y=-2\pi}^{y=\pi}$$

$$= 2\pi+6\pi^2$$

$2\pi+\pi\sin(y)$ is a function that sweeps out horizontal area cross sections from $y=-2\pi$ to $y=\pi$. If we integrate these area cross sections, we accumulate the volume of the solid:



Computing Double Integrals Over a Rectangular Region

Let $z = f(x, y)$ be a surface and let R be the rectangular region in the xy-plane bounded by $a \le x \le b$ and $c \le y \le d$. Calculate $\iint_R f(x,y)\,dA$:

1) Set up $\int_c^d \int_a^b f(x,y)\,dx\,dy$.

2) Hold y constant and integrate with respect to x:

$$\int_c^d \left[F(x,y)\right]_{x=a}^{x=b} dy = \int_c^d \left[F(b,y) - F(a,y)\right] dy$$

3) $F(b,y) - F(a,y)$ is a function of y that measures the area of horizontal cross sections between $y = c$ and $y = d$. So we can integrate these area cross sections to get our volume:

$$\int_c^d \left[F(b,y) - F(a,y)\right] dy$$

1) Set up $\int_a^b \int_c^d f(x,y)\,dy\,dx$.

2) Hold x constant and integrate with respect to y:

$$\int_a^b \left[G(x,y)\right]_{x=c}^{x=d} dx = \int_a^b \left[G(x,d) - G(x,c)\right] dx$$

3) $G(x,d) - G(x,c)$ is a function of x that measures the area of vertical cross sections between $x = a$ and $x = b$. So we can integrate these area cross sections to get our volume:

$$\int_a^b \left[G(x,d) - G(x,c)\right] dx$$

Example 3: Computing Another Double Integral Over a Rectangular Region

Let $f(x,y) = 2\sin(xy) - 1$ and let R be the region in the xy-plane bounded by $x = -4, x = 4, y = -3,$ and $y = 3$. Calculate $\iint_R f(x,y)\,dA$:

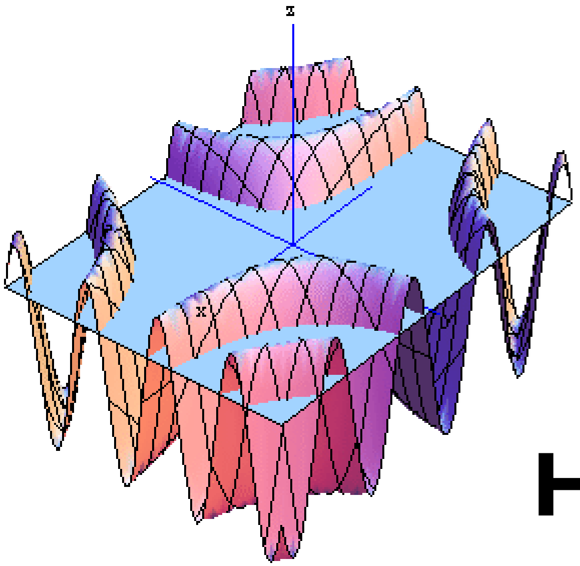
$$\int_{-3}^{3}\int_{-4}^{4}\left(2\sin(xy) - 1\right)dx\,dy = \int_{-3}^{3}\left[-\frac{2\cos(xy)}{y} - x\right]_{x=-4}^{x=4} dy$$

$$= \int_{-3}^{3}\left[\left(-\frac{2\cos(4y)}{y} - 4\right) - \left(-\frac{2\cos(-4y)}{y} - (-4)\right)\right] dy$$

$$= \int_{-3}^{3} -8\,dy$$

$$= \left[-8y\right]_{y=-3}^{y=3}$$

$$= -48$$



How did this end up negative??

Example 4: A Double Integral Over a Non-Rectangular Region

Let $f(x,y) = x^2 + y^2 + 3$ and let R be the region in the xy-plane bounded above by $y = -x^2 + 4$ and below by $y = -5$. Calculate $\iint_R f(x,y)\, dA$:

$$\iint_R f(x,y)\, dA$$

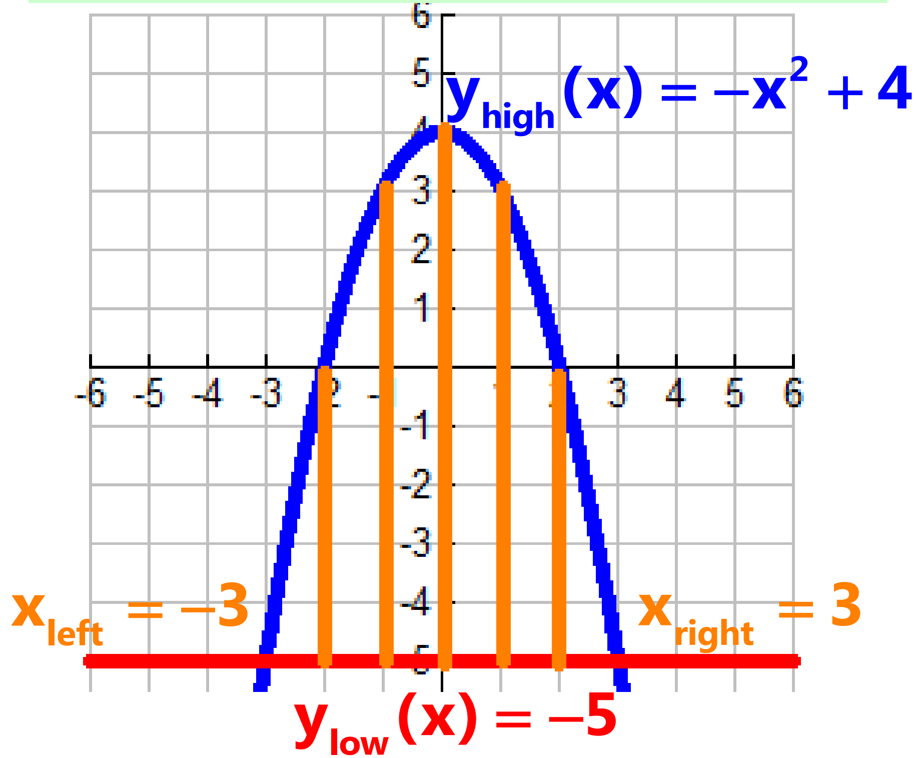
$$= \int_{x_{left}}^{x_{right}} \int_{y_{low}(x)}^{y_{high}(x)} \left(x^2 + y^2 + 3\right) dy\, dx$$

$$= \int_{-3}^{3} \int_{-5}^{-x^2+4} \left(x^2 + y^2 + 3\right) dy\, dx$$

$$= \int_{-3}^{3} \left[yx^2 + \frac{y^3}{3} + 3y \right]_{y=-5}^{y=-x^2+4} dx$$

$$= \int_{-3}^{3} \left(90 - 10x^2 + 3x^4 - \frac{x^6}{3} \right) dx$$

$90 - 10x^2 + 3x^4 - \frac{x^6}{3}$ is a function that sweeps out vertical area cross sections from -3 to 3.



Example 4: A Double Integral Over a Non-Rectangular Region

Let $f(x,y) = x^2 + y^2 + 3$ and let R be the region in the xy-plane bounded above by $y = -x^2 + 4$ and below by $y = -5$. Calculate $\iint\limits_R f(x,y)\,dA$:

If we integrate these cross sections given by

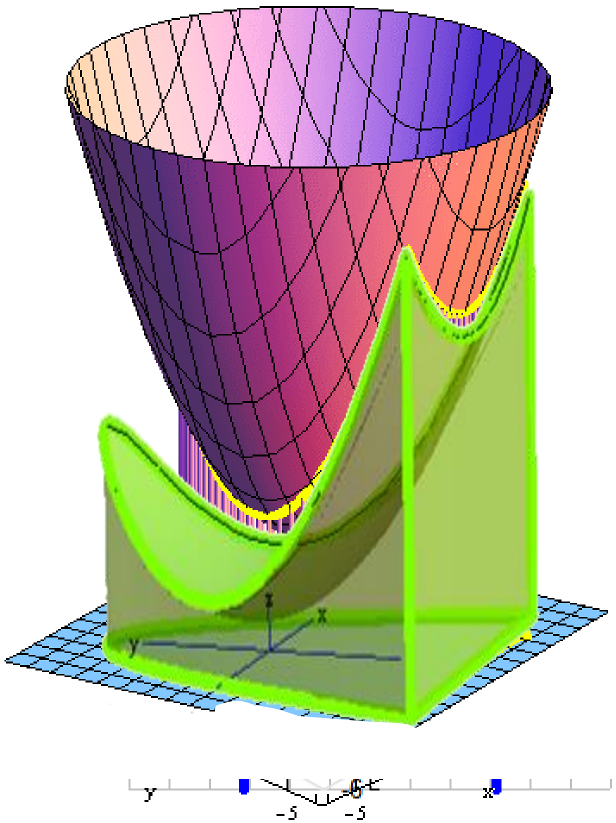
$90 - 10x^2 + 3x^4 - \dfrac{x^6}{3}$ from $x = -3$ to $x = 3$,

we accumulate the volume of the solid:

$$= \int_{-3}^{3} \left(90 - 10x^2 + 3x^4 - \frac{x^6}{3} \right) dx$$

$$= \left[90x - \frac{10}{3}x^3 + \frac{3}{5}x^5 - \frac{x^7}{21} \right]_{x=-3}^{x=3}$$

$$= \frac{15516}{35}$$



Example 5: Change the Order of Integration

Calculate $\iint_R f(x,y)\,dA$ again, but change the order of integration:

$\dfrac{26\sqrt{4-y}}{3}-\dfrac{2}{3}y\sqrt{4-y}+2y^2\sqrt{4-y}$ is a function that sweeps out horizontal area cross sections from $y=-5$ to $y=4$.

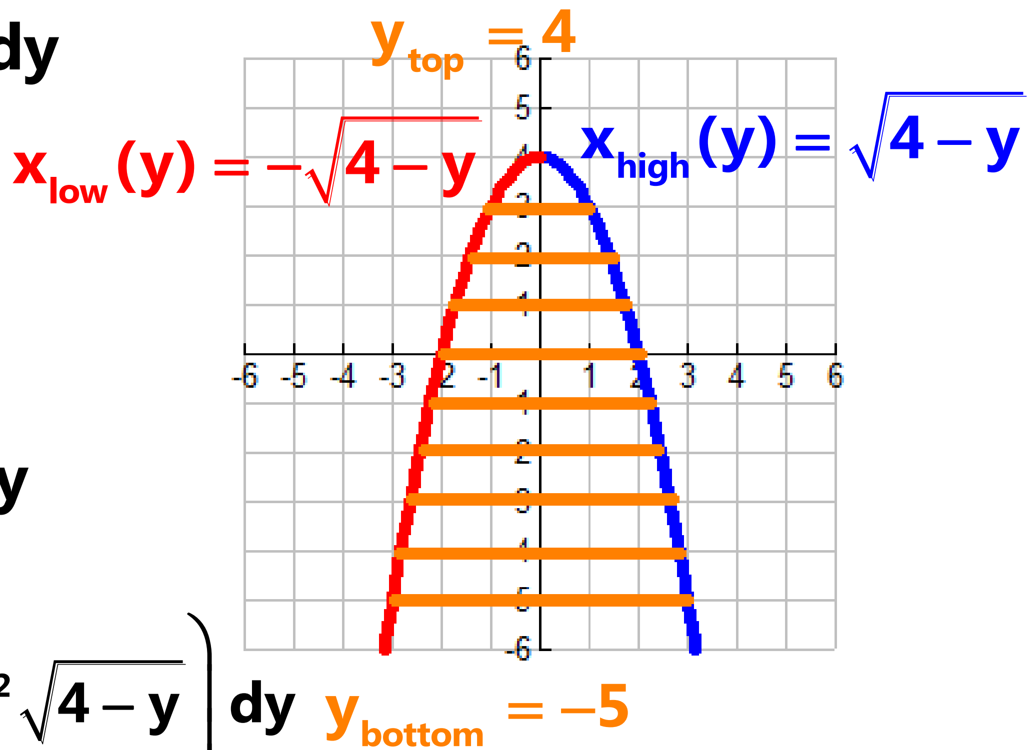
$$\iint_R f(x,y)\,dA$$

$$=\int_{y_{bottom}}^{y_{top}}\int_{x_{low}(y)}^{x_{high}(y)}\left(x^2+y^2+3\right)dx\,dy$$

$$=\int_{-5}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\left(x^2+y^2+3\right)dx\,dy$$

$$=\int_{-5}^{4}\left[\frac{x^3}{3}+y^2x+3x\right]_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}}dy$$

$$=\int_{-5}^{4}\left(\frac{26\sqrt{4-y}}{3}-\frac{2}{3}y\sqrt{4-y}+2y^2\sqrt{4-y}\right)dy$$



Example 5: Change the Order of Integration

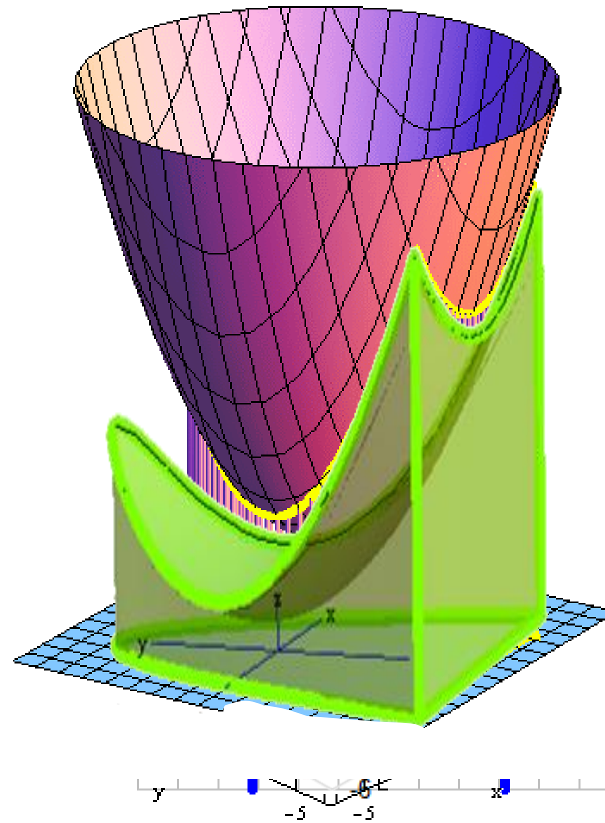
Let $f(x,y) = x^2 + y^2 + 3$ and let R be the region in the xy-plane bounded above by $y = -x^2 + 4$ and below by $y = -5$. Calculate $\iint\limits_R f(x,y)\, dA$:

$\frac{26\sqrt{4-y}}{3} - \frac{2}{3}y\sqrt{4-y} + 2y^2\sqrt{4-y}$ is a function that sweeps out horizontal area cross sections from $y = -5$ to $y = 4$.

$$= \int_{-5}^{4} \left(\frac{26\sqrt{4-y}}{3} - \frac{2}{3}y\sqrt{4-y} + 2y^2\sqrt{4-y} \right) dy$$

$$= \left[-\frac{4}{105}(4-y)^{3/2}\left(15y^2 + 41y + 261\right) \right]_{y=-5}^{y=4}$$

$$= \frac{15516}{35}$$

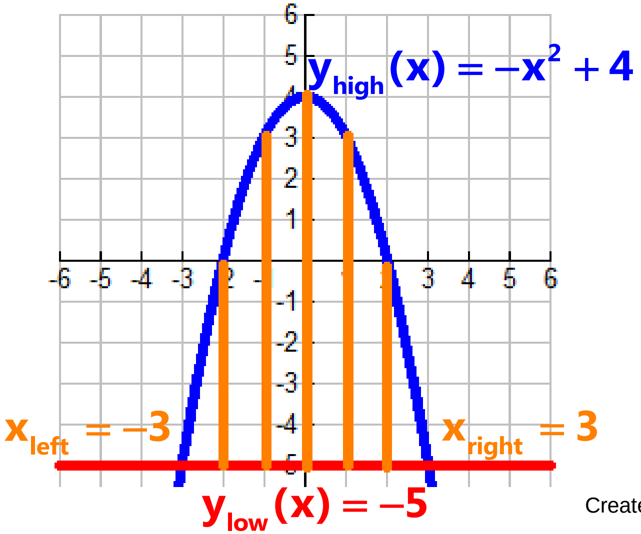


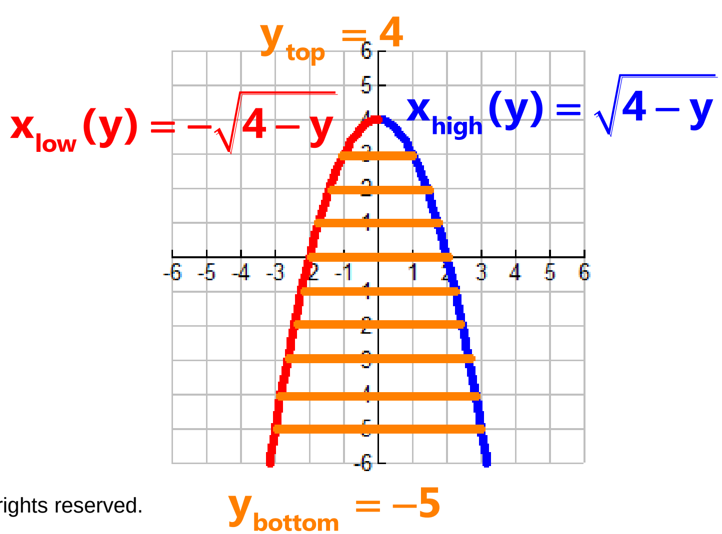
Summary of Example 4 & 5

$$\int_{-3}^{3}\int_{-5}^{-x^2+4}\left(x^2+y^2+3\right)dy\,dx \text{ and } \int_{-5}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\left(x^2+y^2+3\right)dx\,dy \text{ compute}$$

the exact same thing!!!

We have changed the order of integration. This works whenever we can bound the curve with top/bottom bounding functions or left/right bounding functions.





Computing Double Integrals Over a Non-Rectangular Region

Let $z = f(x, y)$ be a surface and let R be the non-rectangular region in the xy-plane bounded by $y = f(x)$ on top and $y = g(x)$ on bottom. Calculate $\iint f(x, y)\, dA$:

1) Set up $\int_{x_{left}}^{x_{right}} \int_{y_{low}(x)}^{y_{high}(x)} f(x, y)\, dy\, dx$

2) Hold x constant and integrate with respect to y:

$$\int_{x_{left}}^{x_{right}} \left[F(x, y)\right]_{y=y_{low}(x)}^{y=y_{high}(x)} dx = \int_{x_{left}}^{x_{right}} \left[F(x, y_{high}(x)) - F(x, y_{low}(x))\right] dx$$

3) $\left[F(x, y_{high}(x)) - F(x, y_{low}(x))\right]$ is a function of x that measures the area of vertical cross sections between $x = x_{left}$ and $x = x_{right}$.

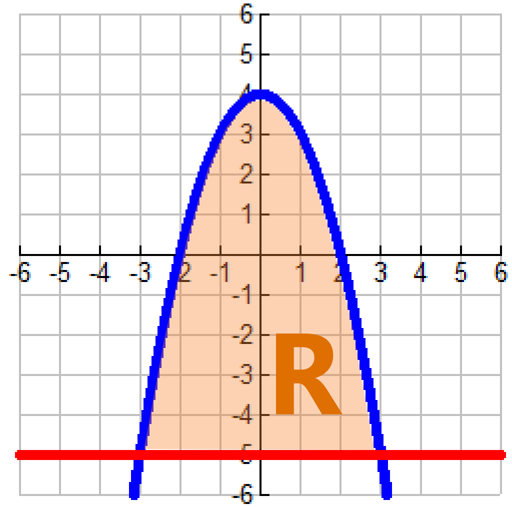
So we can integrate these area cross sections to get our volume:

$$\int_{x_{left}}^{x_{right}} \left[F(x, y_{high}(x)) - F(x, y_{low}(x))\right] dx$$

Likewise for the order integration being reversed.

Example 6: Using the Double Integral to Find the Area of a Region

Let R be the region in the xy-plane from Example 4 and 5 that is bounded above by $y = -x^2 + 4$ and below by $y = -5$. Find the area of R.



Claim: For a region R, $\text{Area}_R = \iint_R 1\, dA$

Old Way:

$$\int_{-3}^{3} \left(\left(-x^2 + 4\right) - \left(-5\right) \right) dx$$

$$= \int_{-3}^{3} \left(-x^2 + 9\right) dx$$

$$= \left[-\frac{x^3}{3} + 9x \right]_{x=-3}^{x=3}$$

$$= 36$$

New Way:

$$\iint_R 1\, dA$$

$$= \int_{-3}^{3} \int_{-5}^{-x^2+4} 1\, dy\, dx$$

$$= \int_{-3}^{3} \left[y \right]_{y=-5}^{y=-x^2+4} dx$$

$$= \int_{-3}^{3} \left(\left(-x^2 + 4\right) - \left(-5\right) \right) dx$$

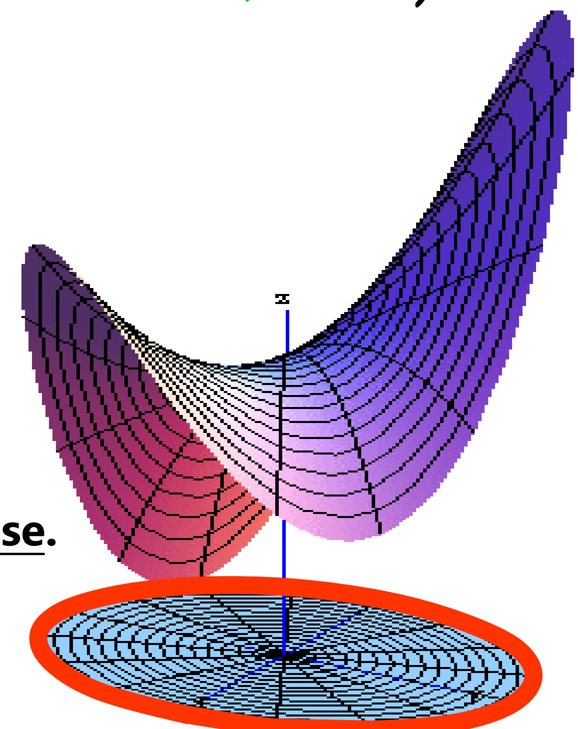
It works!!!

The Gauss-Green Formula

Let R be a region in the xy-plane whose boundary is parameterized by (x(t),y(t)) for $t_{low} \le t \le t_{high}$. Then the following formula holds :

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA = \int_{t_{low}}^{t_{high}} \Big(m\big(x(t), y(t)\big) x'(t) + n\big(x(t), y(t)\big) y'(t) \Big) dt$$

Main Idea: You can compute a double integral as a single integral, as long as you have a reasonable parameterization of the boundary curve of the region R.



Main Rule: For this formula to hold:

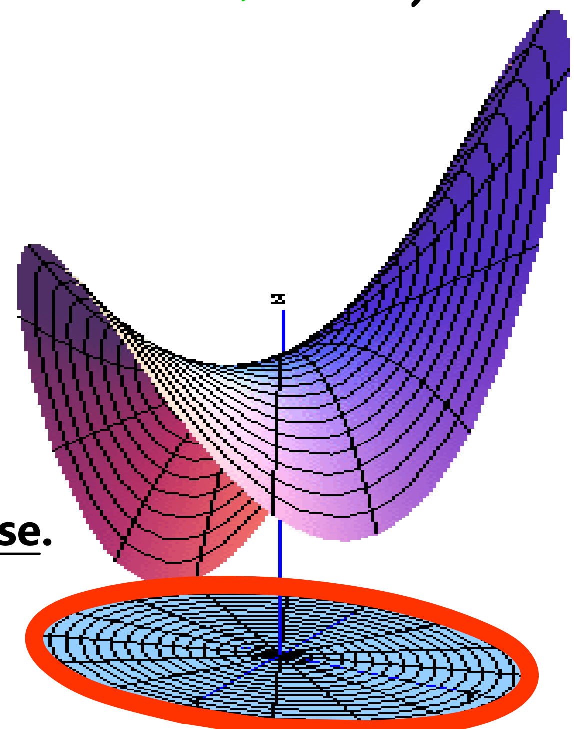
1) Your parameterization must be counterclockwise.
2) Your $t_{low} \le t \le t_{high}$ must bring you around the boundary curve exactly ONCE.
3) Your boundary curve must be a simple closed curve.

The Gauss-Green formula is also known as the <u>Gauss-Green Theorem</u> or simply <u>Green's Theorem</u>

Let R be a region in the xy-plane whose boundary is parameterized by (x(t),y(t)) for $t_{low} \le t \le t_{high}$. Then the following formula holds :

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\Big)dt$$

<u>Main Idea:</u> You can compute a double integral as a single integral, as long as you have a reasonable parameterization of the boundary curve of the region R.



<u>Main Rule:</u> For this formula to hold:

1) Your parameterization must be <u>counterclockwise</u>.
2) Your $t_{low} \le t \le t_{high}$ must bring you around the boundary curve exactly <u>ONCE</u>.
3) Your boundary curve must be a simple closed curve.

Example 6 : Let $f(x,y) = y - xy + 9$ and let R by the region in the xy-plane whose boundary is described by the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$

Find $\iint_R f(x,y)\,dx\,dy$:

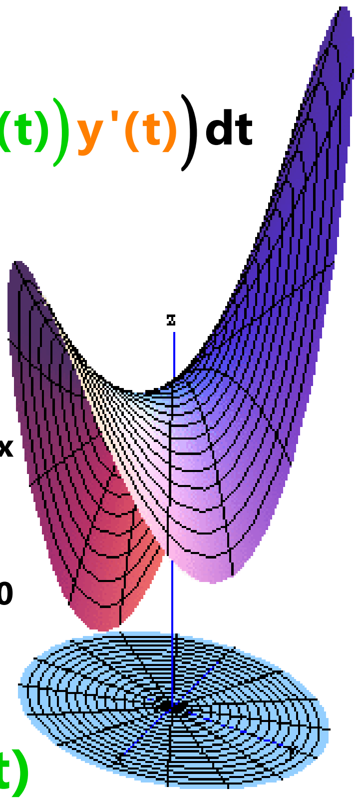
Gauss-Green: $\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy = \int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\Big)dt$

Let $\big(x(t),y(t)\big) = \big(3\cos(t), 4\sin(t)\big)$ for $0 \le t \le 2\pi$

Let $f(x,y) = \frac{\partial n}{\partial x}$ and $0 = \frac{\partial m}{\partial y}$.

Then $n(x,y) = \int_0^x f(s,y)ds = \int_0^x \big(y - sy + 9\big)\,ds = \left[sy - \frac{s^2y}{2} + 9s\right]_0^x$

$$= xy - \frac{x^2y}{2} + 9x$$



So $n\big(x(t),y(t)\big) = 12\sin(t)\cos(t) - 18\cos^2(t)\sin(t) + 27\cos(t)$

Further, $\big(x'(t), y'(t)\big) = \big(-3\sin(t), 4\cos(t)\big)$.

Plug in : $\int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\Big)dt$

Example 6 : Let $f(x,y) = y - xy + 9$ and let R by the region in the xy-plane whose boundary is described by the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$

Plug into Gauss-Green :

$$\int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t) \Big)dt$$

$m\big(x(t),y(t)\big)$	0
$n\big(x(t),y(t)\big)$	$12\sin(t)\cos(t) - 18\cos^2(t)\sin(t) + 27\cos(t)$
$[t_{low}, t_{high}]$	$[0, 2\pi]$
$x'(t)$	$-3\sin(t)$
$y'(t)$	$4\cos(t)$

$$= \int_0^{2\pi} \Big(0 \bullet (-3\sin(t)) + \big(12\sin(t)\cos(t) - 18\cos^2(t)\sin(t) + 27\cos(t)\big)(4\cos(t)) \Big)dt$$

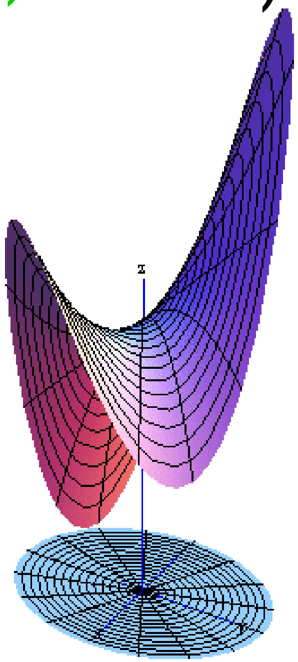
$$= \int_0^{2\pi} \Big(48\cos^2(t)\sin(t) - 72\cos^3(t)\sin(t) + 108\cos^2(t) \Big)dt$$

$$= 108\pi$$

Hint: $\int_a^b \sin^k(t)\cos(t)dt = \frac{\sin^{k+1}(b)}{k+1} - \frac{\sin^{k+1}(a)}{k+1}$

and $\int_a^b \cos^k(t)\sin(t)dt = \frac{-\cos^{k+1}(b)}{k+1} + \frac{\cos^{k+1}(a)}{k+1}$

and $\cos^2(t) = \frac{1+\cos(2t)}{2}$



Process for Using Gauss-Green to Compute Integrals

Let $z = f(x,y)$ be a surface and let R be a region in the xy-plane parameterized by $(x(t),y(t))$ for $t_{low} \le t \le t_{high}$. Calculate $\iint_R f(x,y)\,dA$:

1) Gauss-Green: $\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy = \int_{t_{low}}^{t_{high}} \left(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\right)dt$

2) Substitute as follows:

$m\big(x(t),y(t)\big)$	0
$n\big(x(t),y(t)\big)$	$\int_0^x f(s,y)\,ds$
$[t_{low}, t_{high}]$	Bounds for t in $(x(t),y(t))$
$x'(t)$	Derivative of $x(t)$
$y'(t)$	Derivative of $y(t)$

3) Crunch the integral! Don't forget your trig identities.

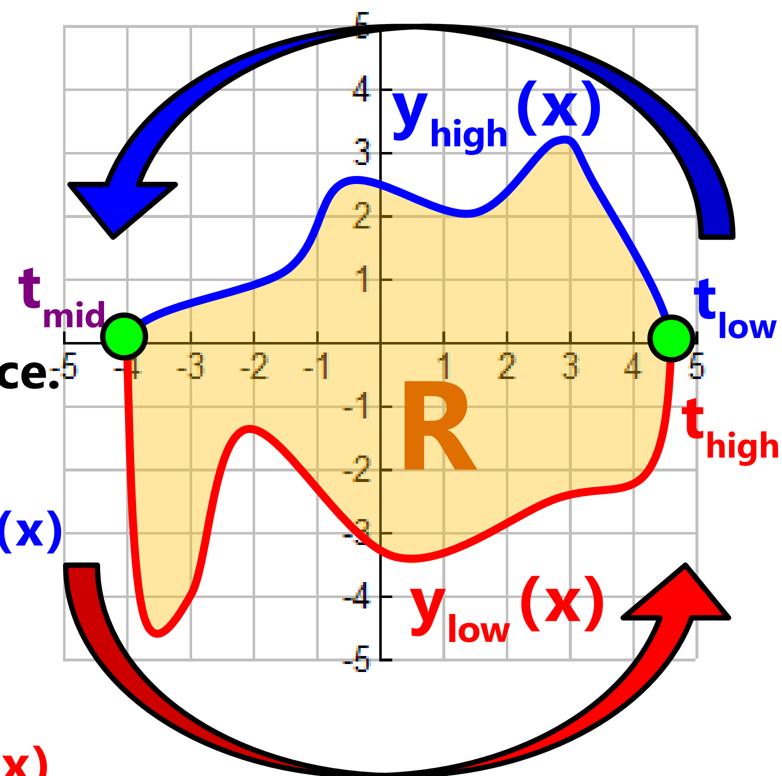
Proof of Gauss-Green Theorem

Prove: $\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\Big)dt$

Let's just prove this for one piece: Show $\iint_R \left(-\frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} \Big(m\big(x(t),y(t)\big)x'(t)\Big)dt$

Let $y_{high}(x)$ be a function of x that traces the boundary of the top half of R and let $y_{low}(x)$ trace the boundary of the bottom half of R.

Let $(x(t), y(t))$ be a counterclockwise parameterization of the boundary of R for $t_{low} < t < t_{high}$ that traverses the boundary only once.



Then $(x(t), y(t))$ for $t_{low} \le t \le t_{mid}$ traces out $y_{high}(x)$ from right to left.

Also $(x(t), y(t))$ for $t_{mid} \le t \le t_{high}$ traces out $y_{low}(x)$ from left to right.

Proof of Gauss-Green Theorem

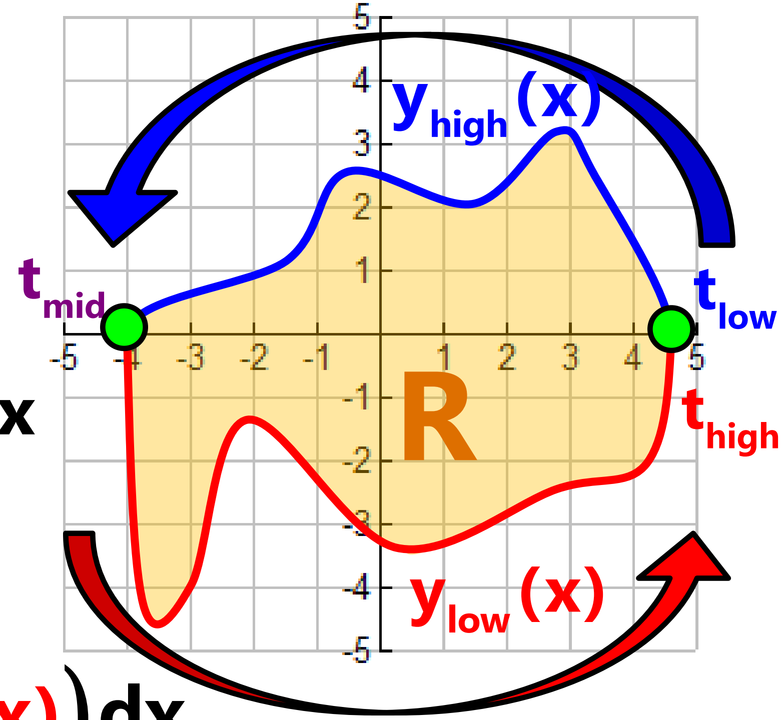
Prove: $$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA = \int_{t_{low}}^{t_{high}} \left(m\big(x(t), y(t)\big) x'(t) + n\big(x(t), y(t)\big) y'(t) \right) dt$$

$$\iint_R \frac{\partial m}{\partial y} \, dA = \int_a^b \int_{y_{low}(x)}^{y_{high}(x)} \frac{\partial m}{\partial y} \, dy \, dx$$

$$= \int_a^b \left[m\big(x, y_{high}(x)\big) - m\big(x, y_{low}(x)\big) \right] dx$$

$$= \int_a^b m\big(x, y_{high}(x)\big) dx - \int_a^b m\big(x, y_{low}(x)\big) dx$$

$$= \int_{t_{mid}}^{t_{low}} m\big(x(t), y(t)\big) \frac{dx}{dt} dt - \int_{t_{mid}}^{t_{high}} m\big(x(t), y(t)\big) \frac{dx}{dt} dt$$



Proof of Gauss-Green Theorem

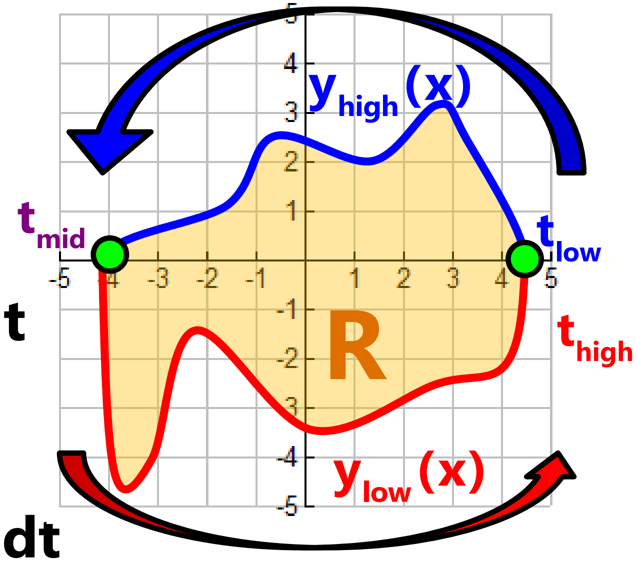
Prove: $$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} \left(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\right)dt$$

$$= \int_{t_{mid}}^{t_{low}} m\big(x(t),y(t)\big)\frac{dx}{dt}dt - \int_{t_{mid}}^{t_{high}} m\big(x(t),y(t)\big)\frac{dx}{dt}dt$$

$$= \int_{t_{mid}}^{t_{low}} m\big(x(t),y(t)\big)x'(t)dt - \int_{t_{mid}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$$

$$= -\int_{t_{low}}^{t_{mid}} m\big(x(t),y(t)\big)x'(t)dt - \int_{t_{mid}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$$

$$= -\int_{t_{low}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$$



Hence $$\iint_R \left(-\frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$$

Proof of Gauss-Green Theorem

Prove: $\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} \left(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\right)dt$

Hence $\iint_R \left(-\frac{\partial m}{\partial y}\right) dA = \int_{t_{low}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$

We can likewise prove... $\iint_R \left(\frac{\partial n}{\partial x}\right) dA = \int_{t_{low}}^{t_{high}} \left(n\big(x(t),y(t)\big)y'(t)\right)dt$

Put it Together :

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dA = \iint_R \frac{\partial n}{\partial x} dA - \iint_R \frac{\partial m}{\partial y} dA$$

$$= \int_{t_{low}}^{t_{high}} n\big(x(t),y(t)\big)y'(t)dt + \int_{t_{low}}^{t_{high}} m\big(x(t),y(t)\big)x'(t)dt$$

$$= \int_{t_{low}}^{t_{high}} \left(m\big(x(t),y(t)\big)x'(t) + n\big(x(t),y(t)\big)y'(t)\right)dt$$

When Should I Use What??

$\int_{y_{bottom}}^{y_{top}} \int_{x_{low}(y)}^{x_{high}(y)} f(x,y)\,dx\,dy$	$\int_{x_{left}}^{x_{right}} \int_{y_{low}(x)}^{y_{high}(x)} f(x,y)\,dy\,dx$	$\int_{t_{low}}^{t_{high}} n\big(x(t),y(t)\big)\,y'(t)\,dt$
Integrate with respect to x first then y when you have a region R that is bounded on the left and right by two functions of y:	Integrate with respect to y first then x when you have a region R that is bounded on the top and bottom by two functions of x:	Use Gauss-Green when you have a region R whose boundary you can parameterize with a single function of t, (x(t),y(t)) for $t_{low} \le t \le t_{high}$.