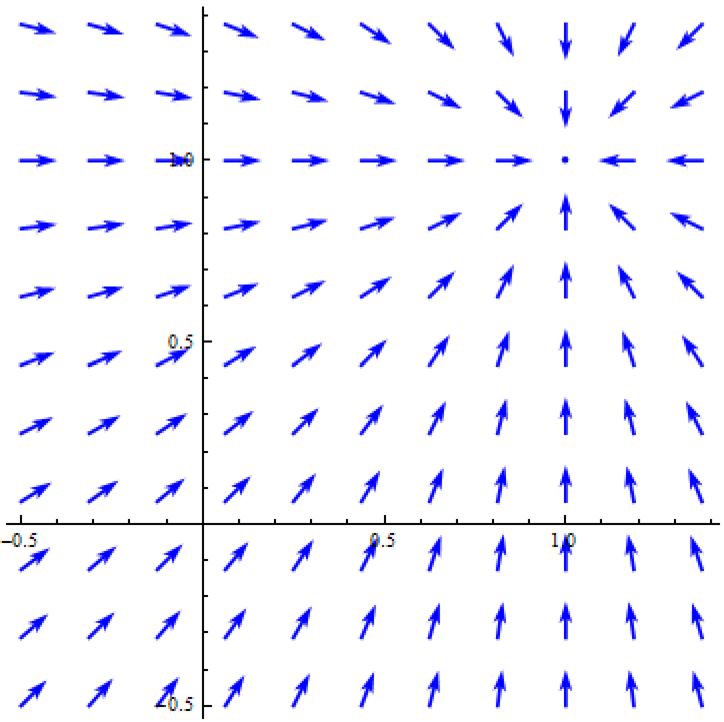
## Lesson 6

# Vector Fields Acting on a Curve

# Example 1: What is a Vector Field?

**<u>Definition:</u> A vector field assigns to each point in the plane, (x,y), a vector, $\left(m(x,y), n(x,y)\right)$.**

**<u>Example 1:</u> Plot the vector field $\left(-x+1, -y+1\right)$.**




**The best way to imagine this is to pretend the xy-plane is a body of water with currents and whirlpools or natural springs that add or take away water.**

**The direction of the arrow shows you where the current is pushing you and the magnitude of the vector tells you how strong the current is.**

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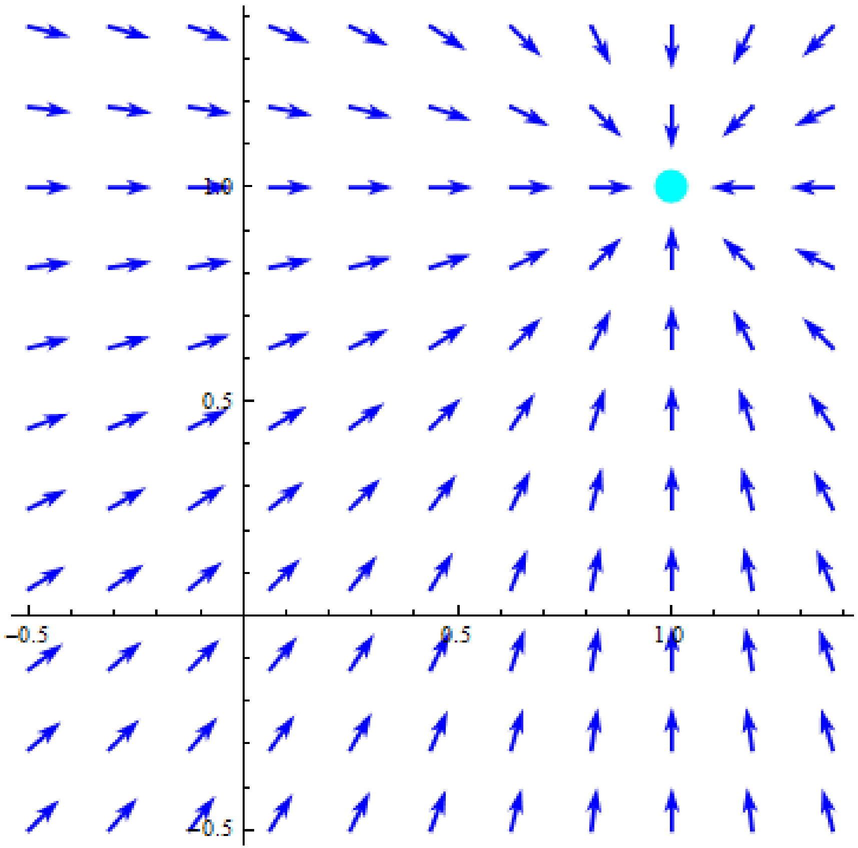
# Example 1: What is a Vector Field?

**Example 1:** Plot the vector field $(-x+1,-y+1)$.

**If you were swimming in this ocean, what would you experience?**

**In your opinion, what is the most notable point on this plot? Why?**



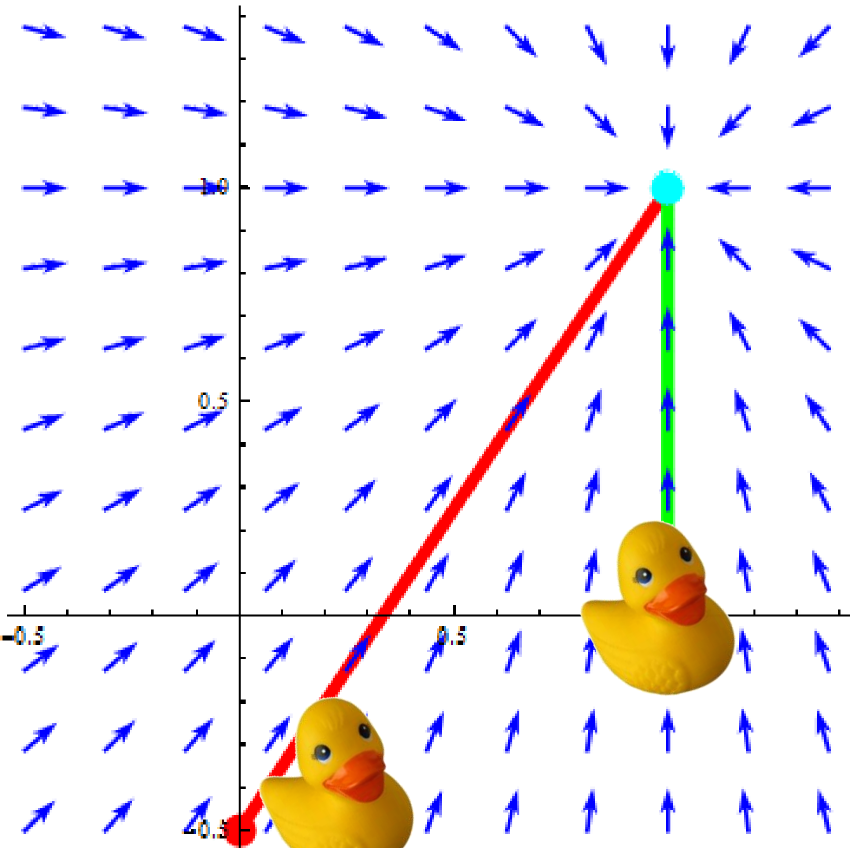

**(1,1) is called a sink.**

**All of the vectors in a neighborhood around (1,1) flow into it.**

# Example 1: What is a Vector Field?

**Example 1:** **Plot the vector field $\left(-x+1,-y+1\right)$.**

**The vector field plot maps the currents in this ocean. If we drop a rubber ducky at any point on this plot, where will it go?**




**These paths are called trajectories.**

**A trajectory, (x(t),y(t)), of a vector field, Field(x,y), is a path that follows the vector field at any given point.**

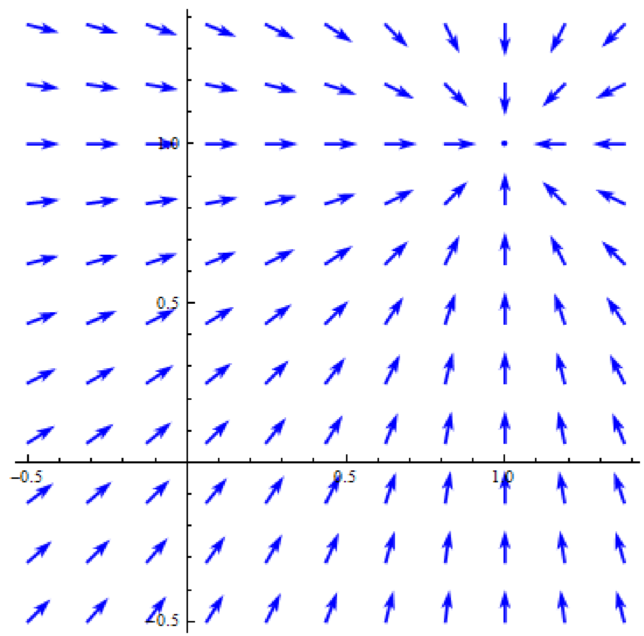
**More precisely, a trajectory is a path, (x(t),y(t)), such that the tangent vectors, to the path are the field vectors, $(x'(t), y'(t)) = \text{Field}(x(t), y(t))$. That is, the field vectors whose tails are on the trajectory are tangent to the trajectory.**

# Definition: Vector Field

**We define a <u>vector field</u> to be an assignment of one vector to each point in the plane. That is, for every point (x,y) in the plane, there is a vector associated with it, given by:**

$$\left(m(x,y), n(x,y)\right)$$

<u>**Example :**</u> $\left(m(x,y), n(x,y)\right) = \left(-x+1, -y+1\right)$

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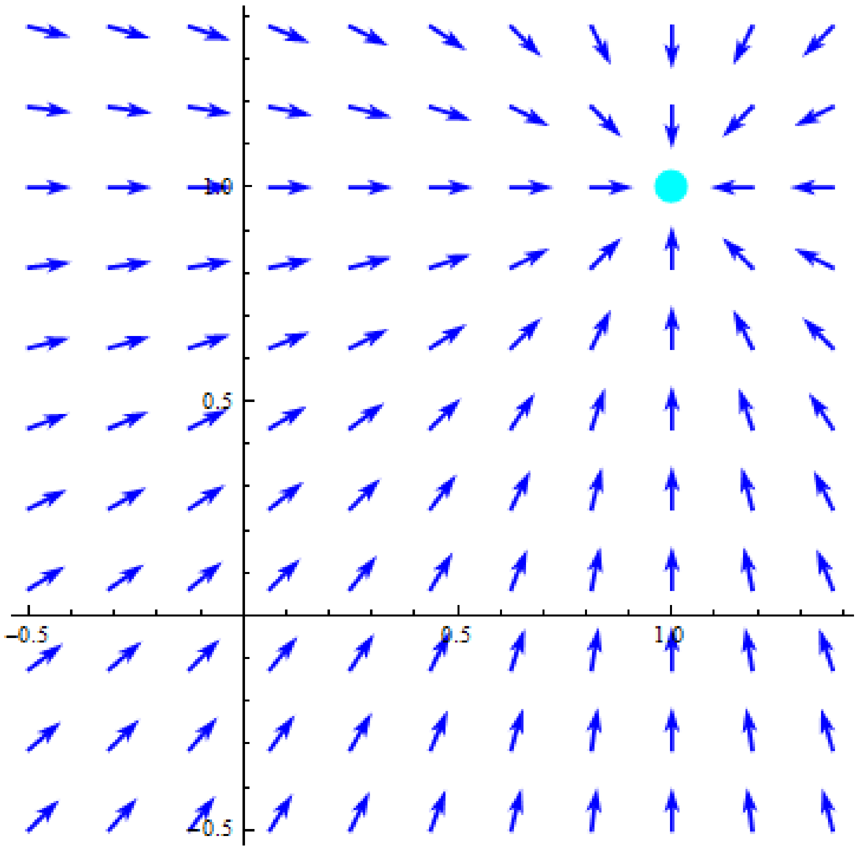
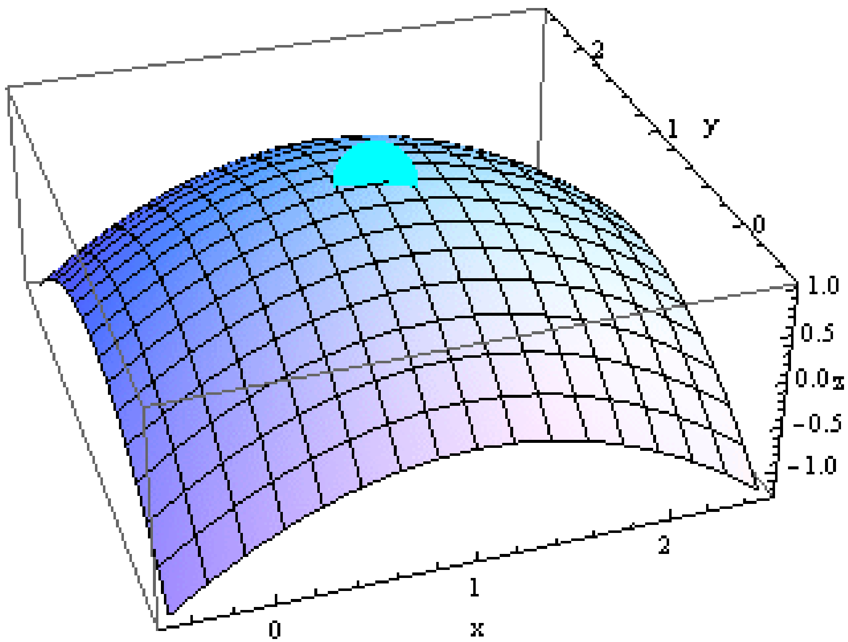
# Example 1: What is a Vector Field?

**Example 1:** What is the relationship between the vector field $\left(-x+1, -y+1\right)$ and the surface $f(x,y) = -\frac{x^2}{2} - \frac{y^2}{2} + x + y$?



# Example 1: What is a Vector Field?

**Example 1:** **What is the relationship between the vector field $\left(-x+1,-y+1\right)$ and the surface $f(x,y)=-\frac{x^2}{2}-\frac{y^2}{2}+x+y$?**

**That's right!**

**Answer Below This Box…**

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# Something to Think About...

**What are some ways one could tell if**

$$\left(m(x,y),n(x,y)\right)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$$

**for some $z=f(x,y)$?**

**We'll address this next chapter, but it is a question you are certainly equipped to answer today...**

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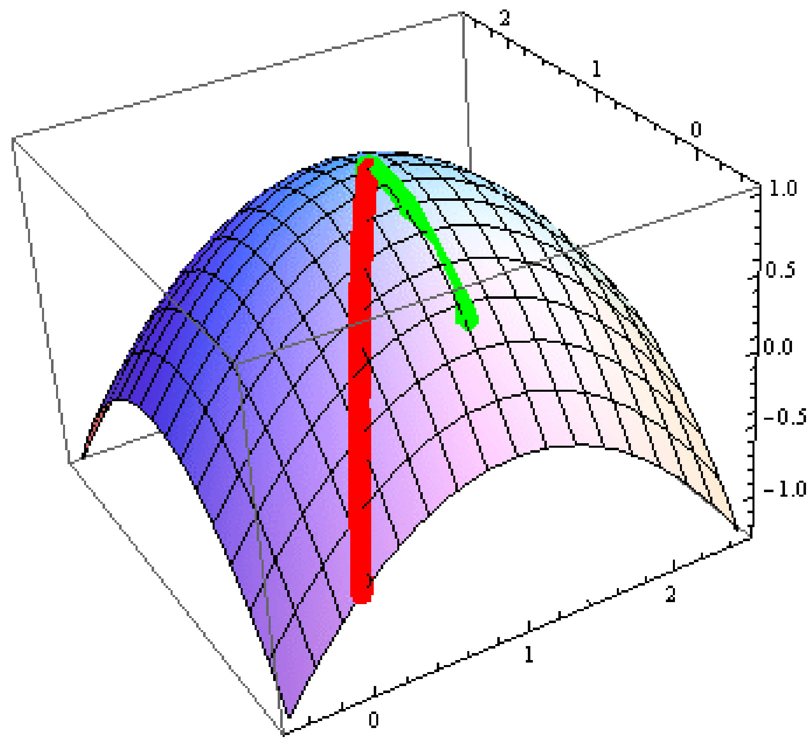
# Example 1: What is a Gradient Field?

**Example 1: What do our trajectories on the vector field mean in the context of the surface** $f(x,y) = -\frac{x^2}{2} - \frac{y^2}{2} + x + y$**?**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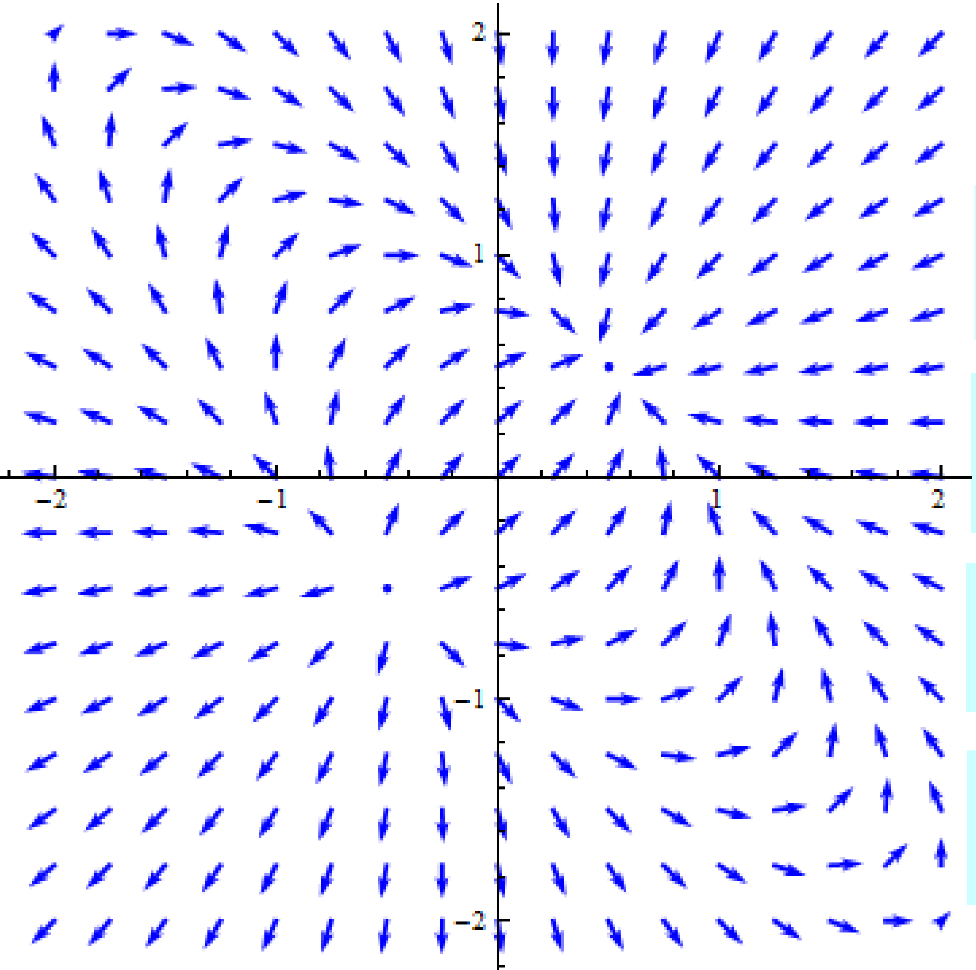
# Example 2: Another Gradient Field

**Example 2:** **Plot the gradient field for** $f(x,y) = \dfrac{x+y}{e^{x^2+y^2}}$ **:**



$$\nabla f = e^{-x^2-y^2}\left(1-2x(x+y), 1-2y(x+y)\right)$$

**How many points stand out to you in this vector field?**

**This vector field has a source and a sink.**

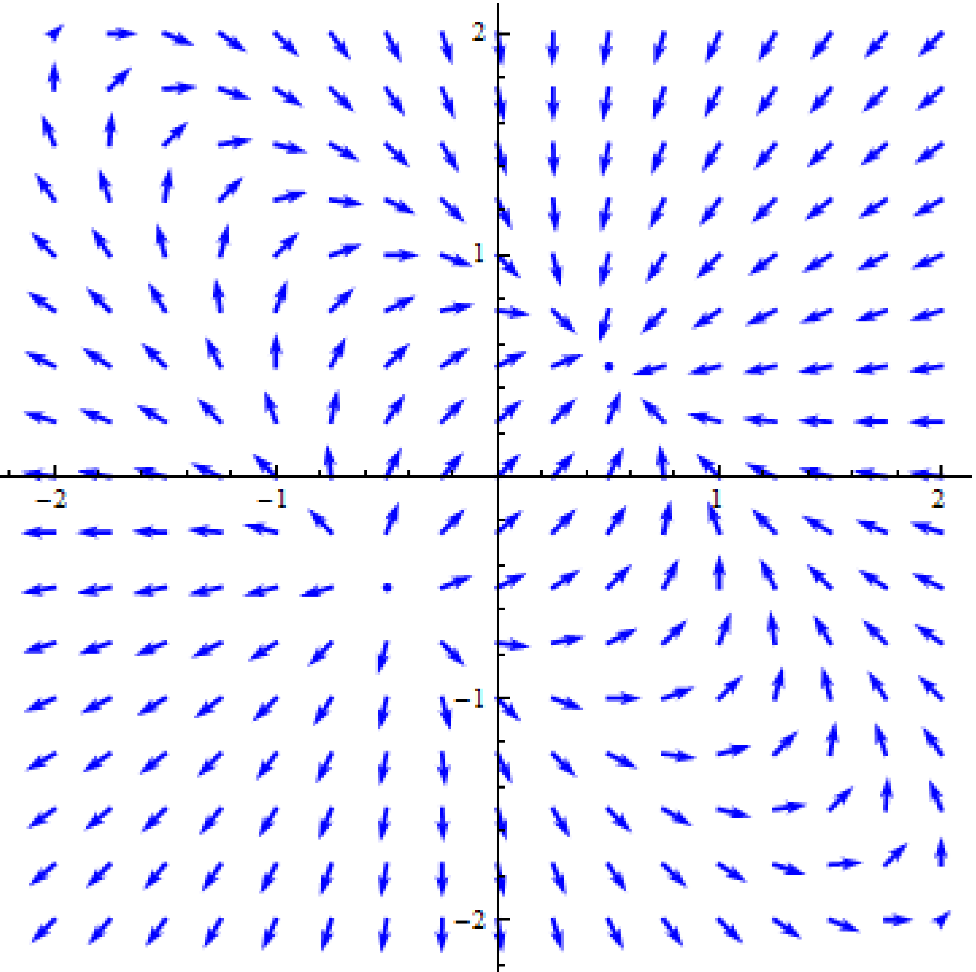
**What do the source and sink correspond to on the surface?**

**Source $\rightarrow$ Local Minimum**

**Sink $\rightarrow$ Local Maximum**

# Example 2: Another Gradient Field

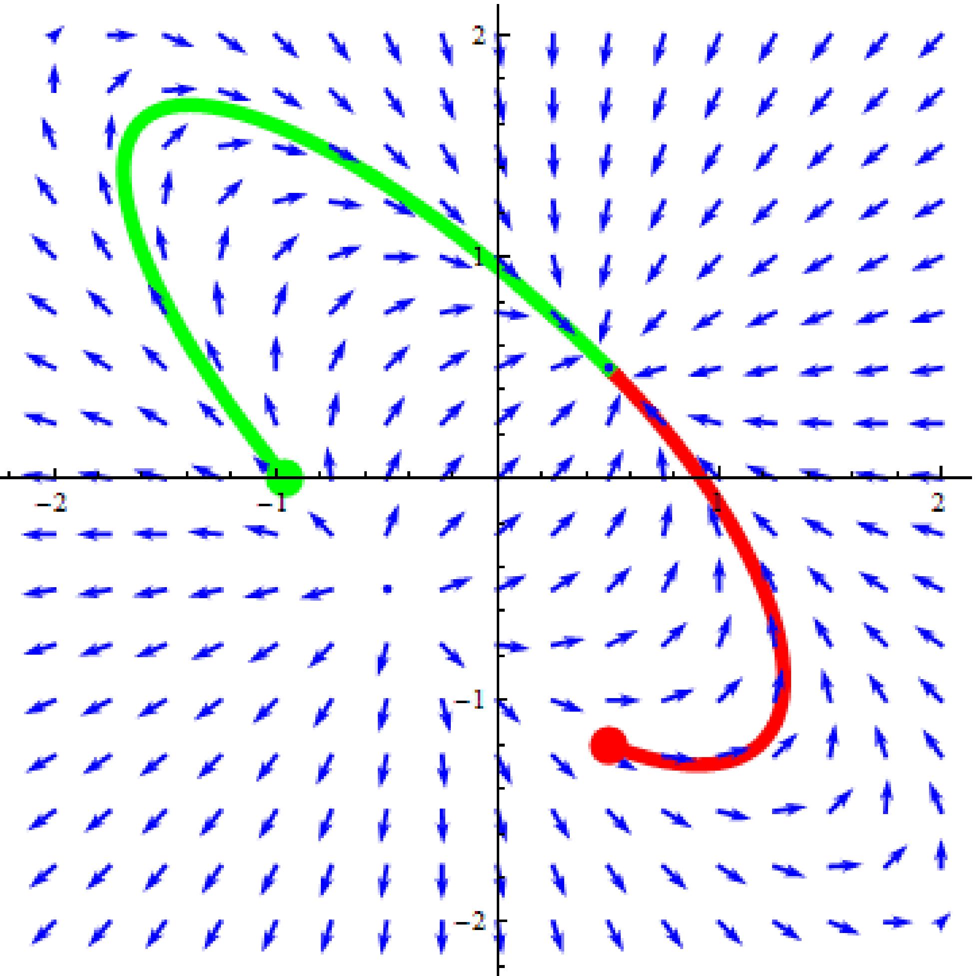
**Example 2:** **Plot the gradient field for** $f(x,y) = \dfrac{x+y}{e^{x^2+y^2}}$ **:**




**I picked the equation for f(x,y) so that I could guarantee it had a global maximum and a global minimum somewhere on it. How did I know?**

# Example 2: Another Gradient Field

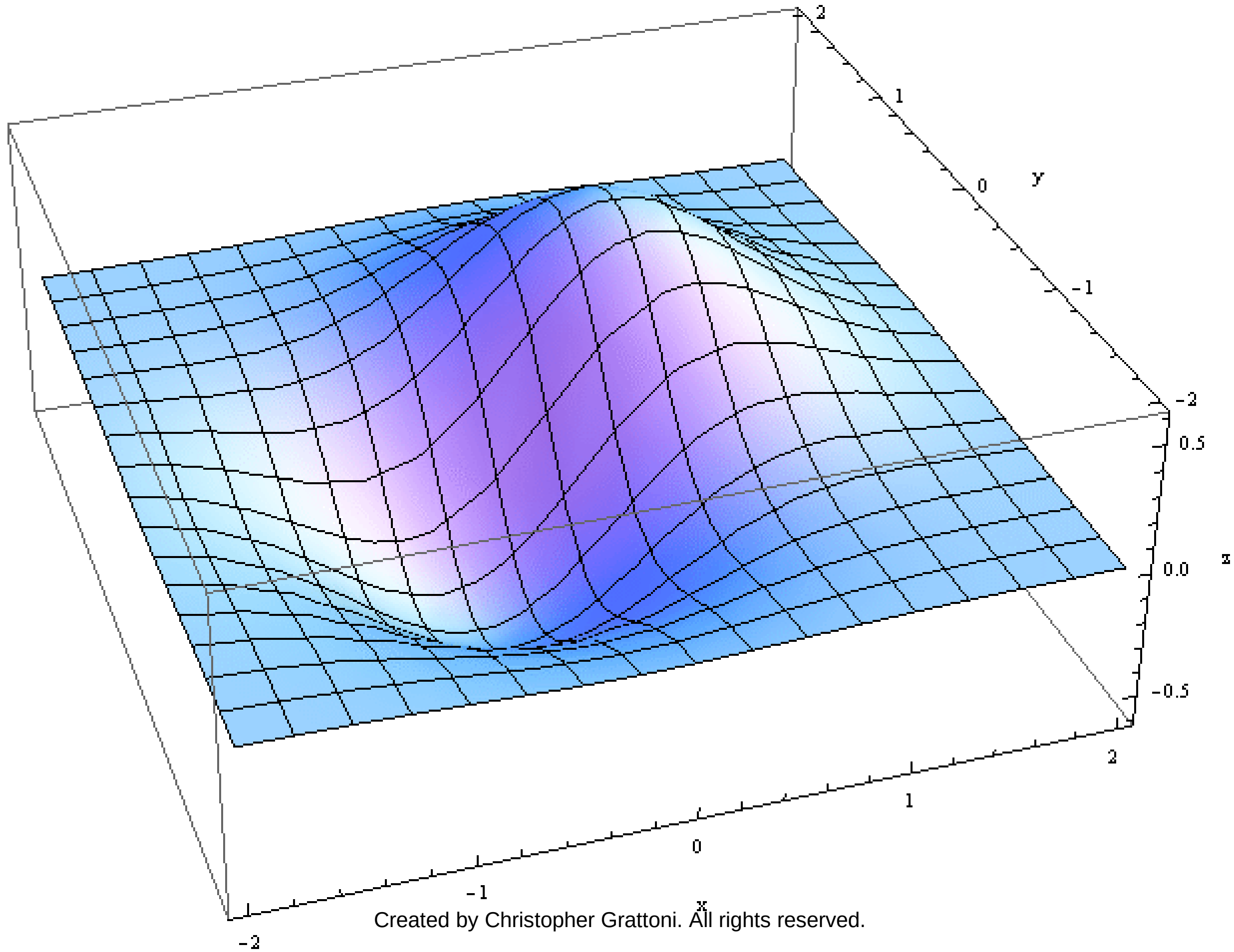
**Example 2:** Plot the gradient field for $f(x,y) = \dfrac{x+y}{e^{x^2+y^2}}$ :




Let's plot two trajectories :

# Example 2: Another Gradient Field

**Example 2:** Now plot the actual surface $f(x,y) = \dfrac{x+y}{e^{x^2+y^2}}$ :

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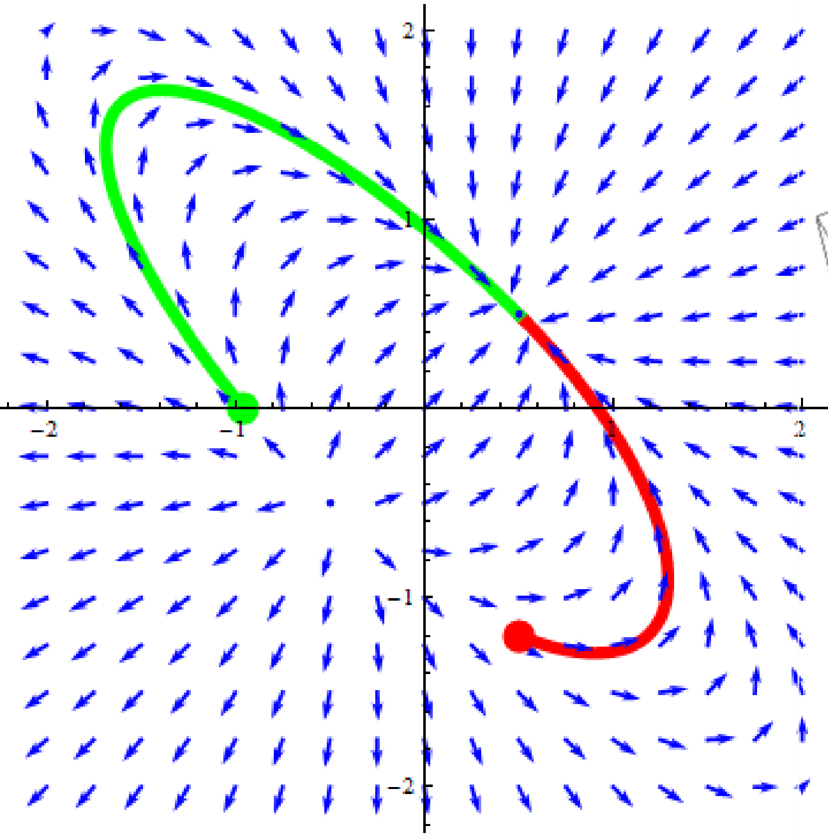
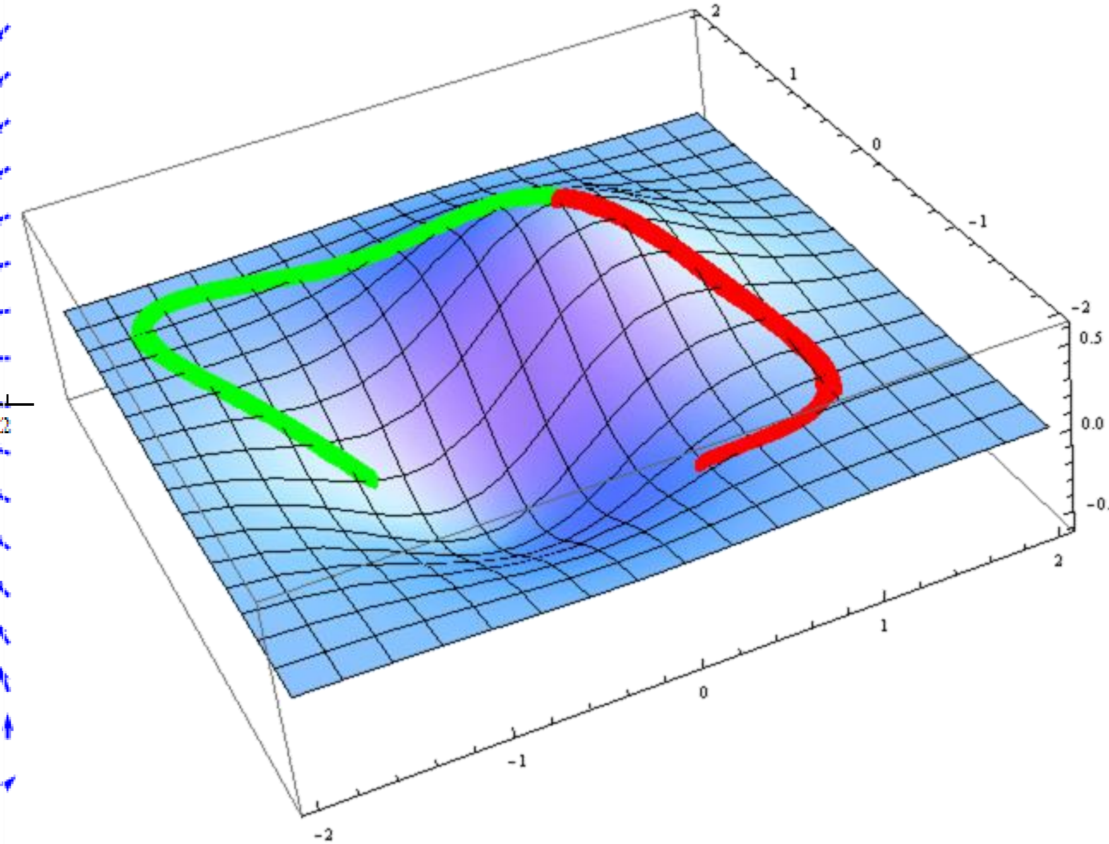
# Example 2: Another Gradient Field

**Example 2:** Finally, compare the gradient field and $f(x, y) = \dfrac{x+y}{e^{x^2+y^2}}$:

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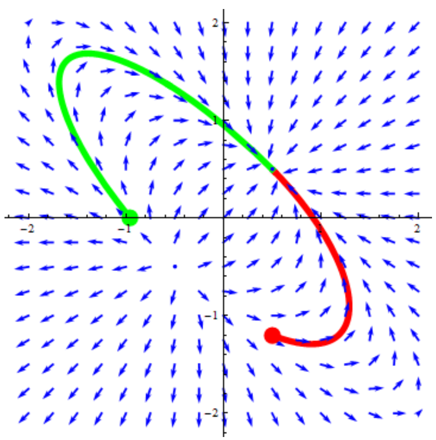
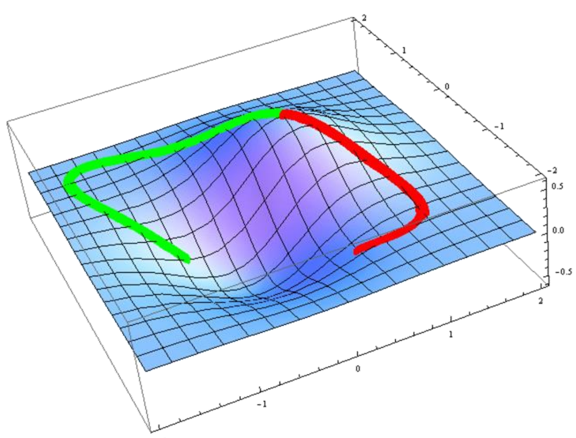
# Definition: Gradient Field

We define a gradient field to be a vector field $\left(m(x,y), n(x,y)\right)$ such that there exists a function $z = f(x,y)$ such that

$$\left(m(x,y), n(x,y)\right) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Gradient fields have the property that there is a surface associated with them. Sources correspond with minima and sinks correspond with maxima.





**WARNING : In order to be classified as a gradient field, your vector field must not have any points (x,y) where $\left(m(x,y), n(x,y)\right)$ is undefined.**

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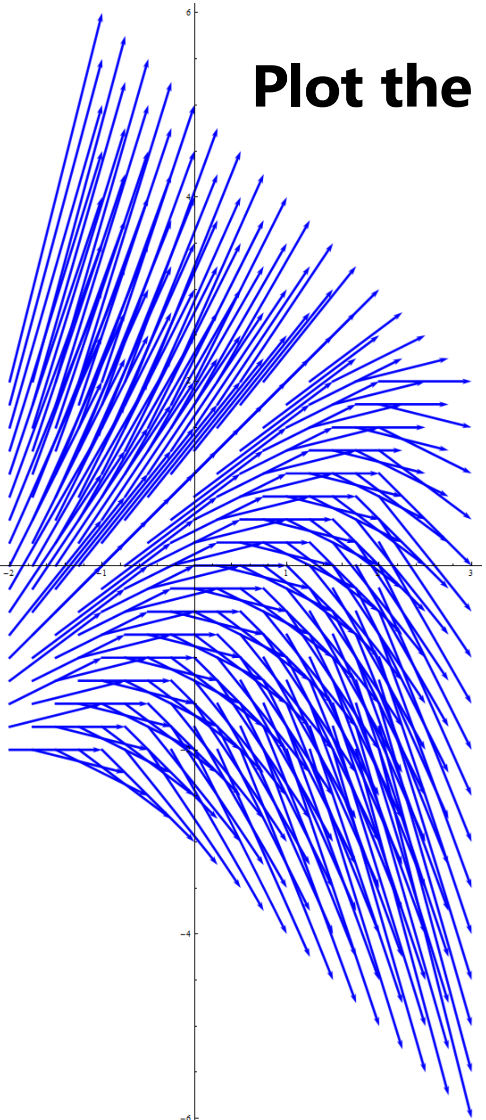
# Example 3: A Slope Field is a Kind of Vector Field



Plot the slope field for $\dfrac{dy}{dx} = y - x$.

This is equivalent to the vector field $\left(1, y - x\right)$.

Why...?

Why doesn't this look like a slope field from last year?

Hint: What is the vector at (0,10)?

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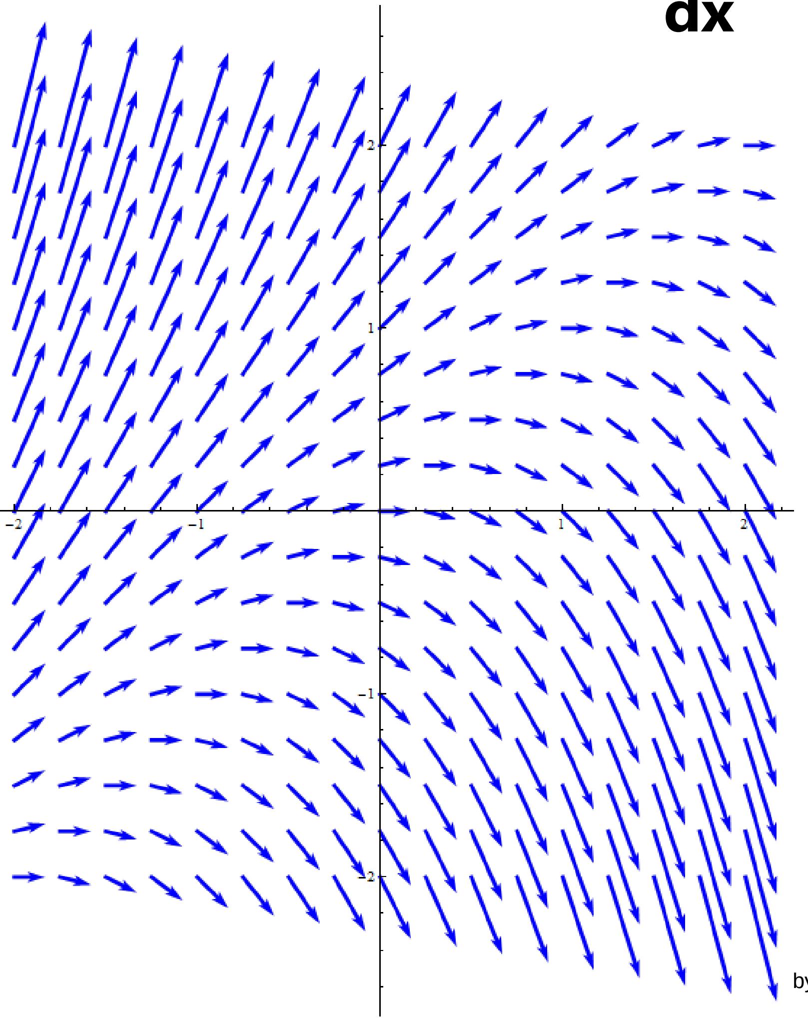
# Example 3: A Slope Field is a Kind of Vector Field

**Plot the slope field for $\dfrac{dy}{dx} = y - x$.**




by Christoph

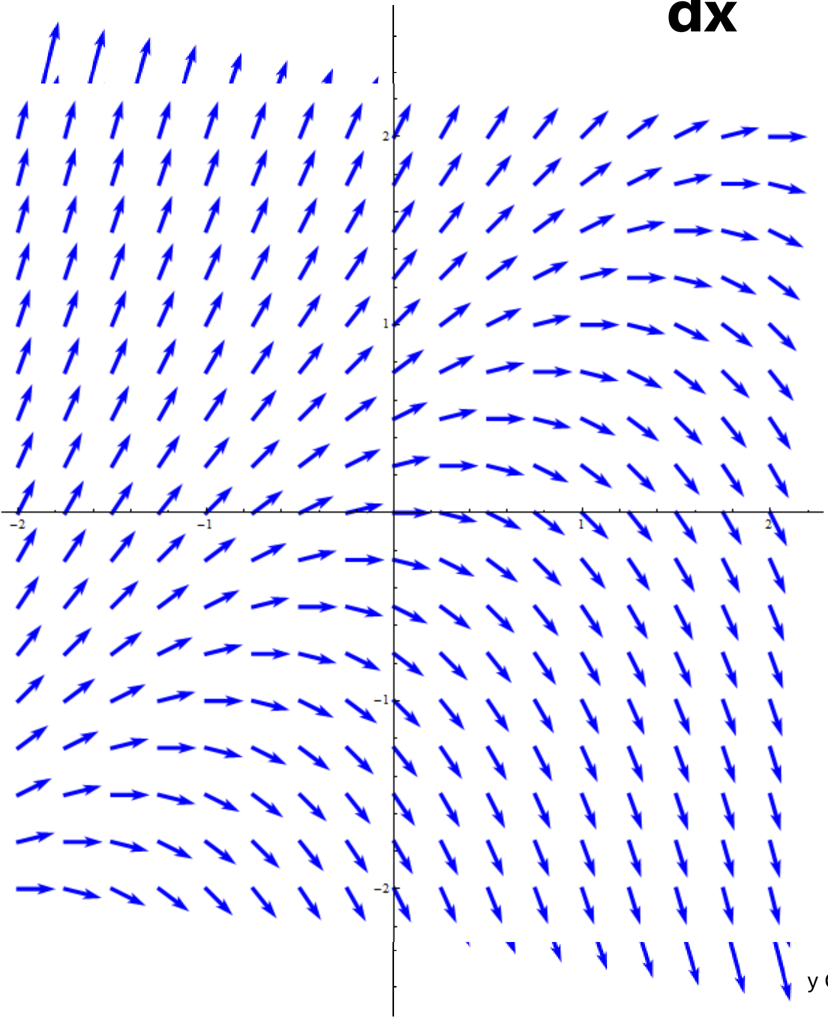
**The problem is that $\left(1, y - x\right)$ is an <u>unscaled</u> vector field.**

**We can scale it by whatever factor makes it more useable. We will try $0.17\left(1, y - x\right)$:**

**A scaled vector field still has direction AND magnitude information, but it's a lot less cluttered.**

# Example 3: A Slope Field is a Kind of Vector Field

**Plot the slope field for $\dfrac{dy}{dx} = y - x$.**




**Even more familiar would be for us to normalize each vector and scale it appropriately:**

$$\frac{1}{\sqrt{(1, y-x)\bullet(1, y-x)}}(1, y-x)$$

**This gets rid of all magnitude information, but it is what we were used to looking at last year.**

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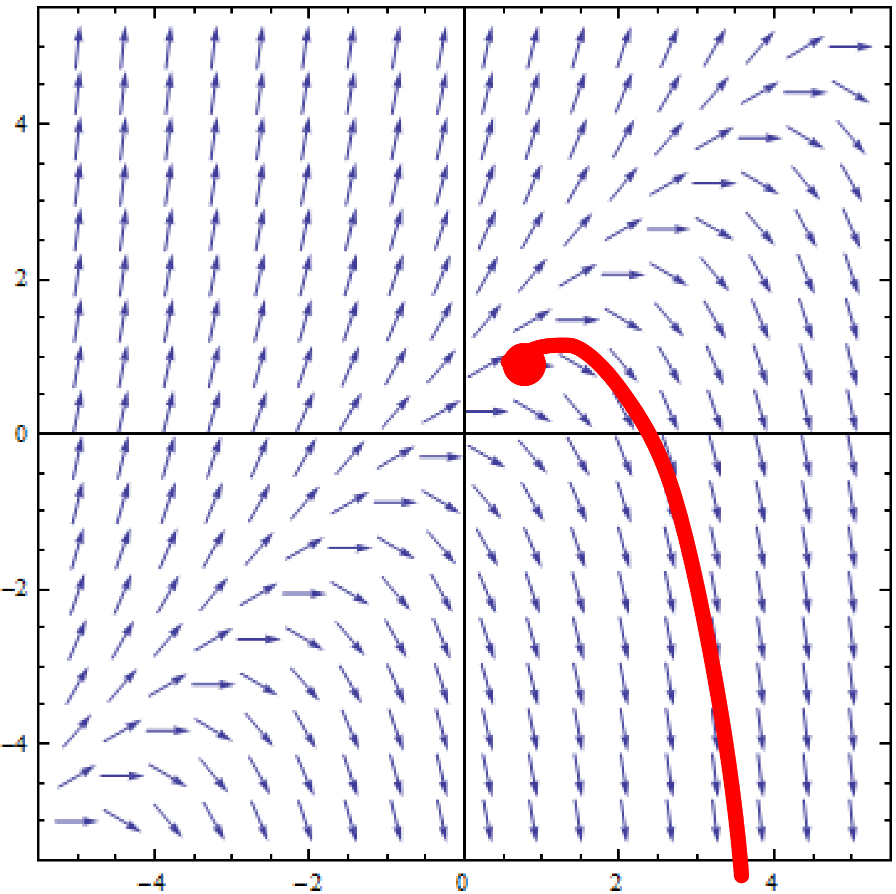
# Example 3: A Slope Field is a Kind of Vector Field

**Plot a trajectory on the slope field for $\dfrac{dy}{dx} = y - x$ through (1,1).**

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**What does this trajectory represent in terms of the differential equation $\dfrac{dy}{dx}$?**

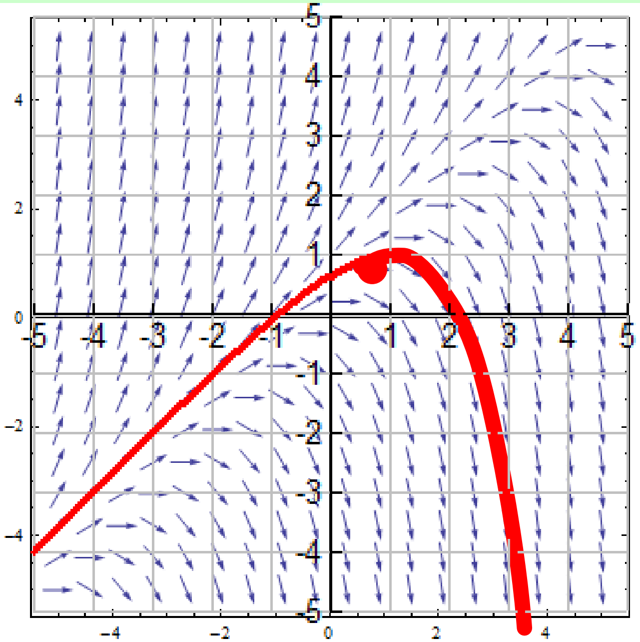
**This trajectory is an approximation of a particular solution to $\dfrac{dy}{dx}$ through (1,1).**

**We would use this when we don't know how to (or can't) integrate the $\dfrac{dy}{dx}$ given.**

# Example 3: A Slope Field is a Kind of Vector Field

**Plot a trajectory on the slope field for $\frac{dy}{dx} = y - x$.**

**We don't know how to integrate $\frac{dy}{dx} = y - x$ because the differential equation isn't separable… A trajectory is our only real choice…**



**I can integrate it for verification purposes:**

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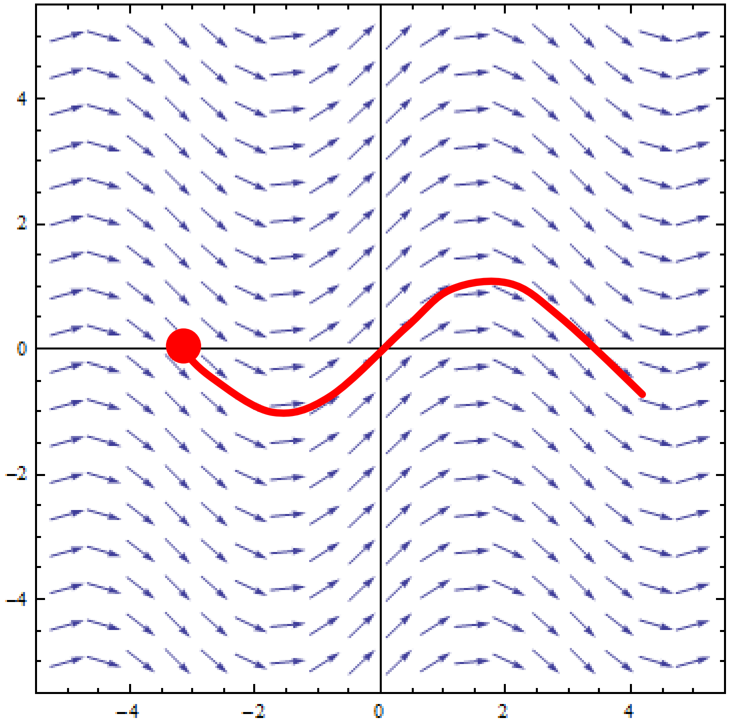
$$\mathbf{y = -e^{x-1} + x + 1}$$

# Example 4: One More Slope Field

**Plot the slope field for $y' = \cos(x)$ and the trajectory through $(-\pi, 0)$.**




**What function does our trajectory through $(-\pi, 0)$ look like?**

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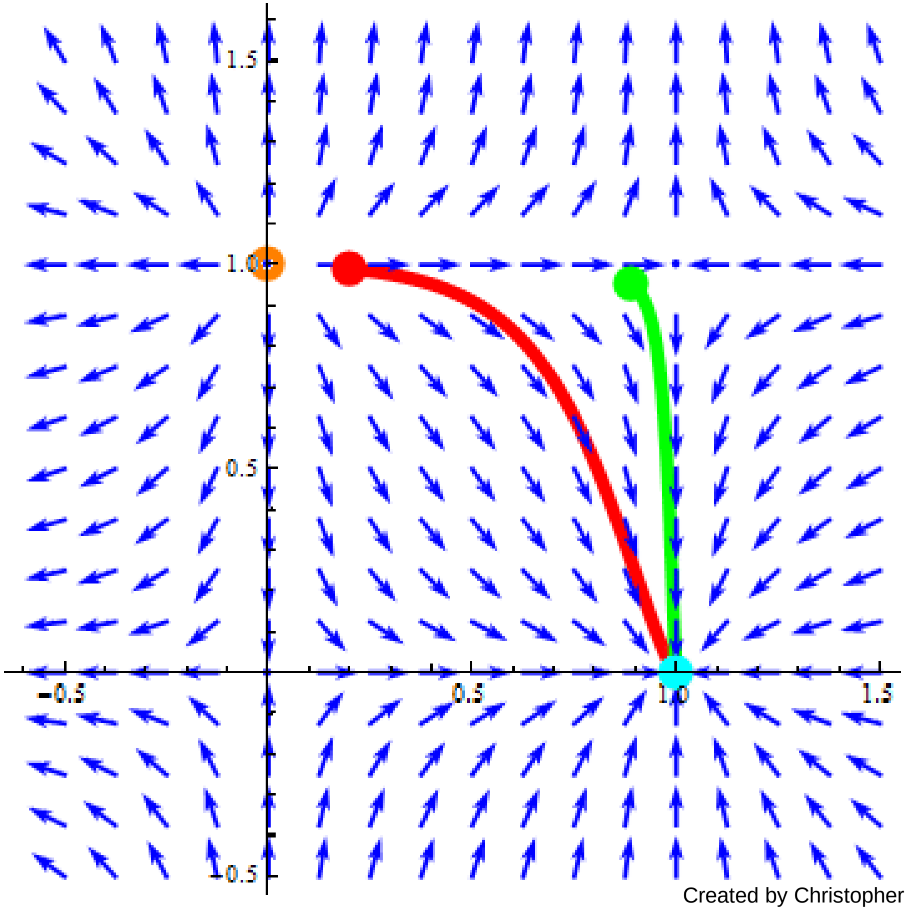
# Example 5: A General Vector Field

**Plot the vector field given by $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$.**

**Then plot a couple of trajectories.**




**This vector field is neither a slope field nor a gradient field. Some vector fields are just… vector fields.**

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# Vector Fields Venn Diagram

## Vector Fields

$$\left(m(x,y), n(x,y)\right)$$

### Slope Fields

$$\left(1, y'(x,y)\right)$$

### Gradient Fields

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

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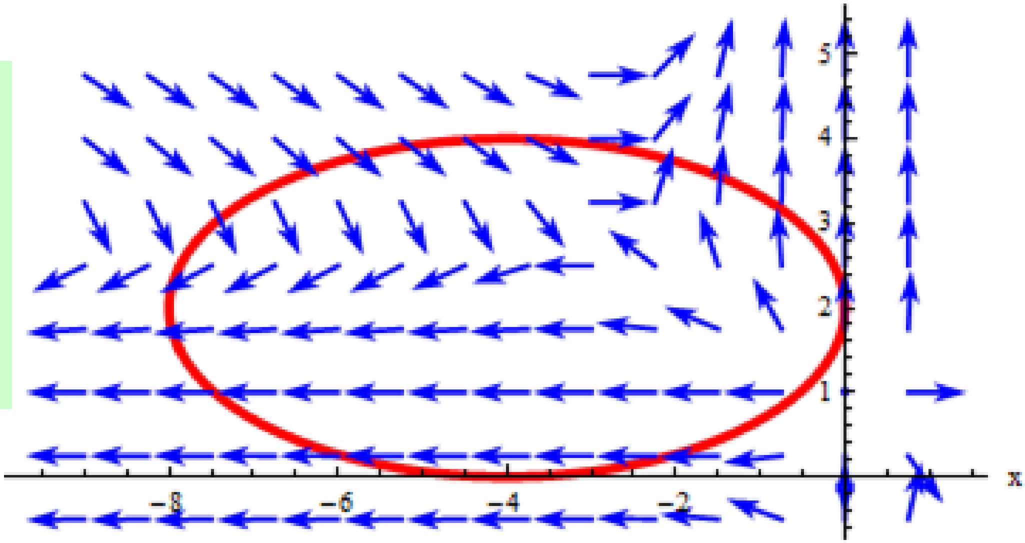
# Example 6: A Vector Field Acting on a Curve

**Consider the vector field from Ex 5: $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$. We'll let this vector field act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$**

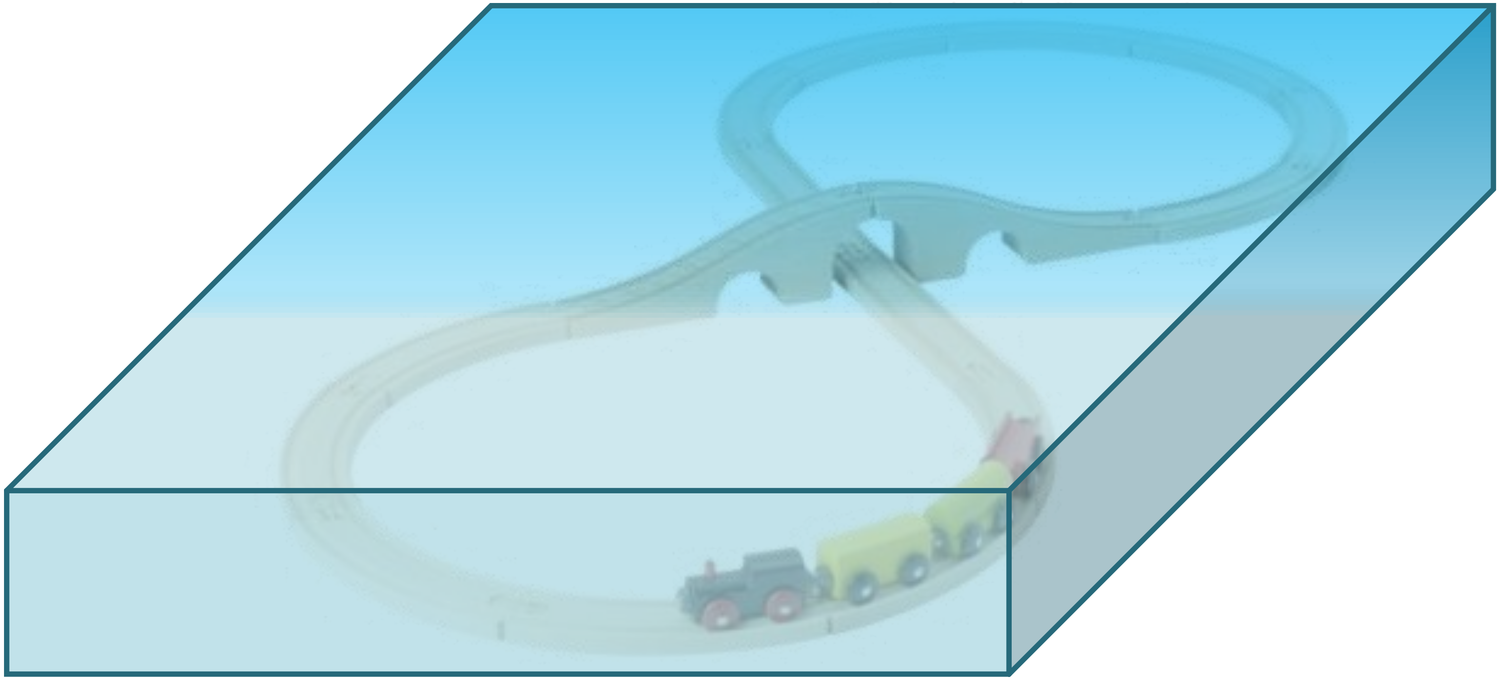
**We are placing a curve into the vector field and watching how points moving along the curve experience this...**




**Note that the BLUE VECTOR FIELD and the RED CURVE are two distinct mathematical objects. The red curve is NOT a trajectory of the vector field. We are putting these two objects together to see how the blue one affects the red one.**

# A Vector Field Acting on a Curve

**It's like putting a toy train set into a rushing whitewater river and watching how the water affects the train that is stuck on its track.**



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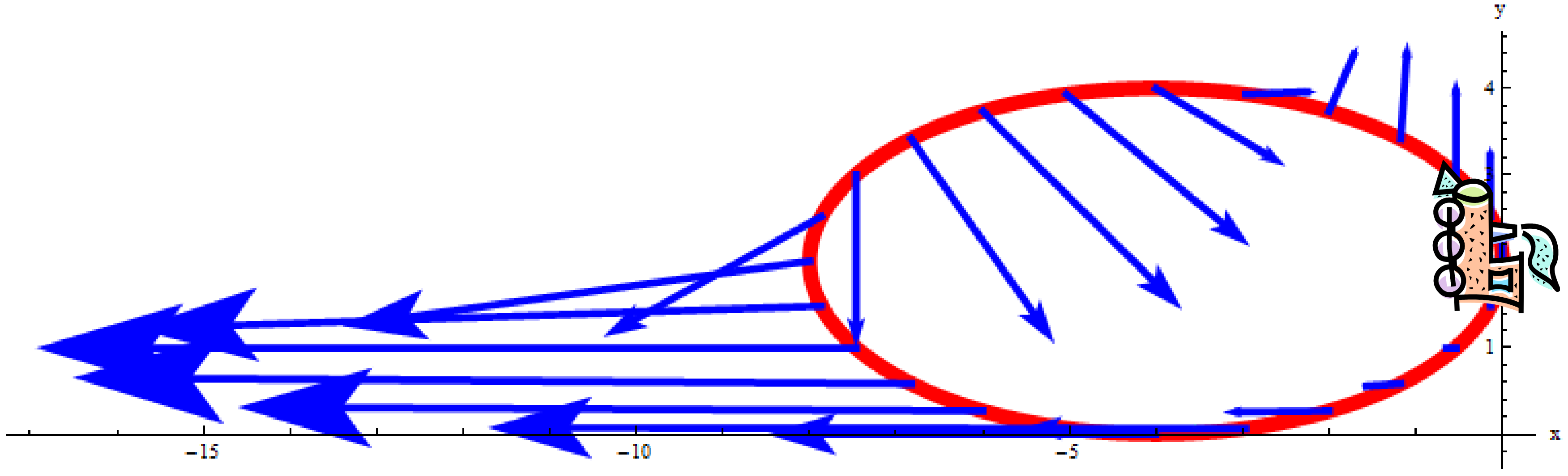
**Example 6 :** Let the vector field $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$ act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$.

We really don't need to plot the WHOLE vector field, just the vectors whose tails are on the ellipse:




How would you experience a loop around this curve on a train?

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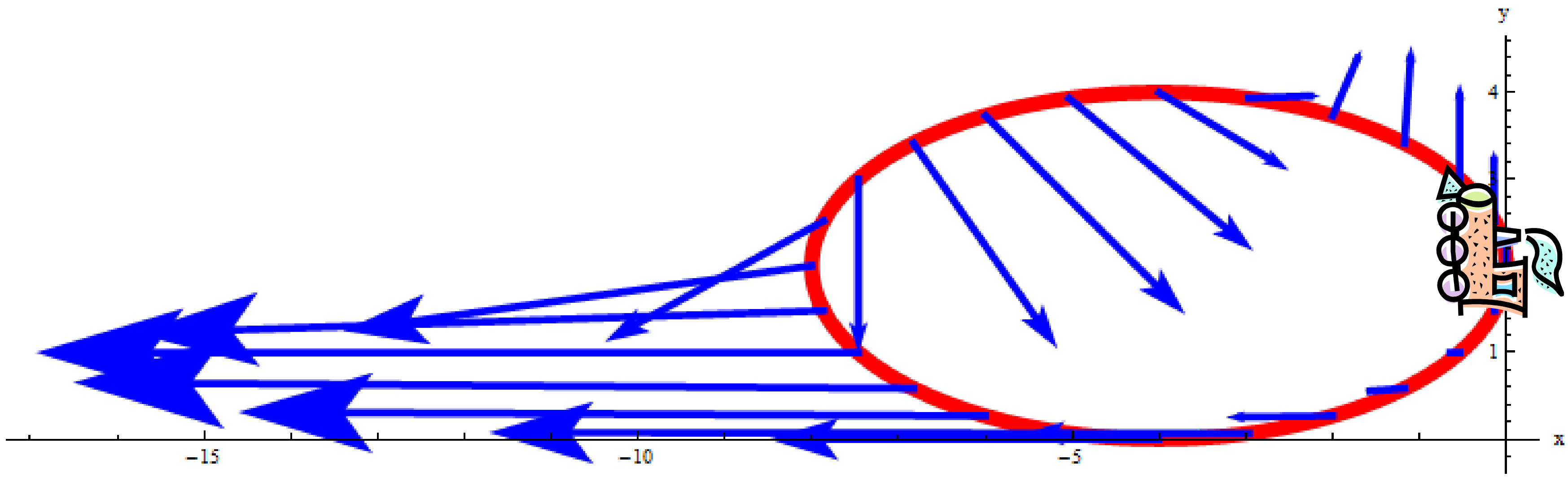
**Example 6**: Let the vector field $\left((3x^2 - 3x)(y-3), (3y^2 - 3y)(x+3)\right)$ act on the ellipse $(x(t), y(t)) = (-4, 2) + (4\cos(t), 2\sin(t))$.

## Describe what the train experiences here:




## Our summary:

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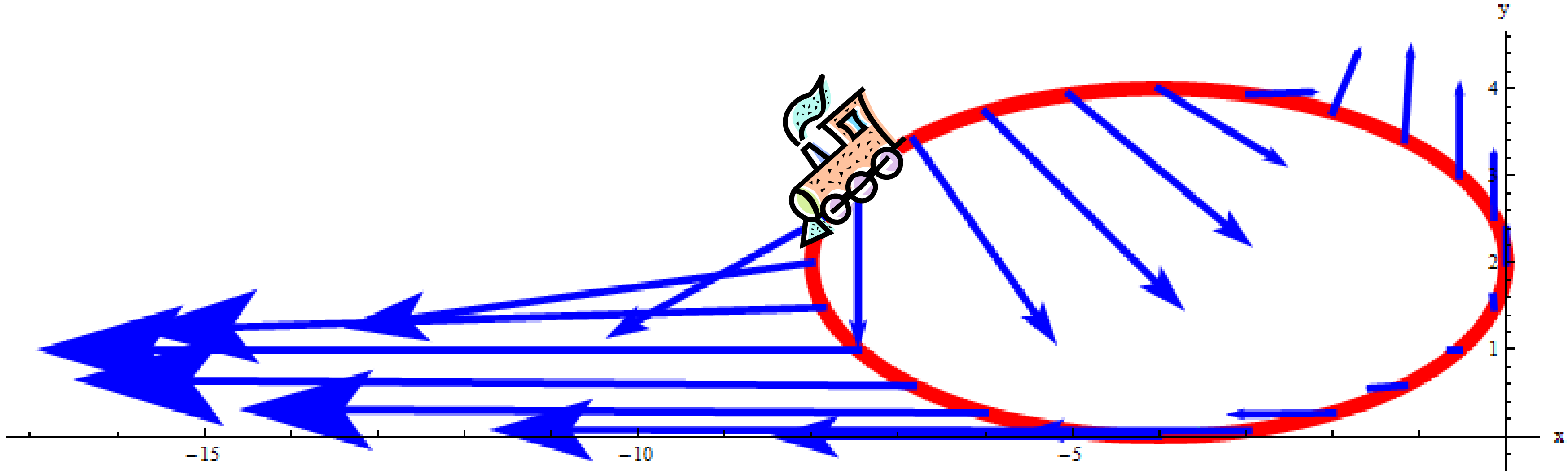
## Example 6 : Let the vector field $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$ act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$.

**Describe what the train experiences here:**

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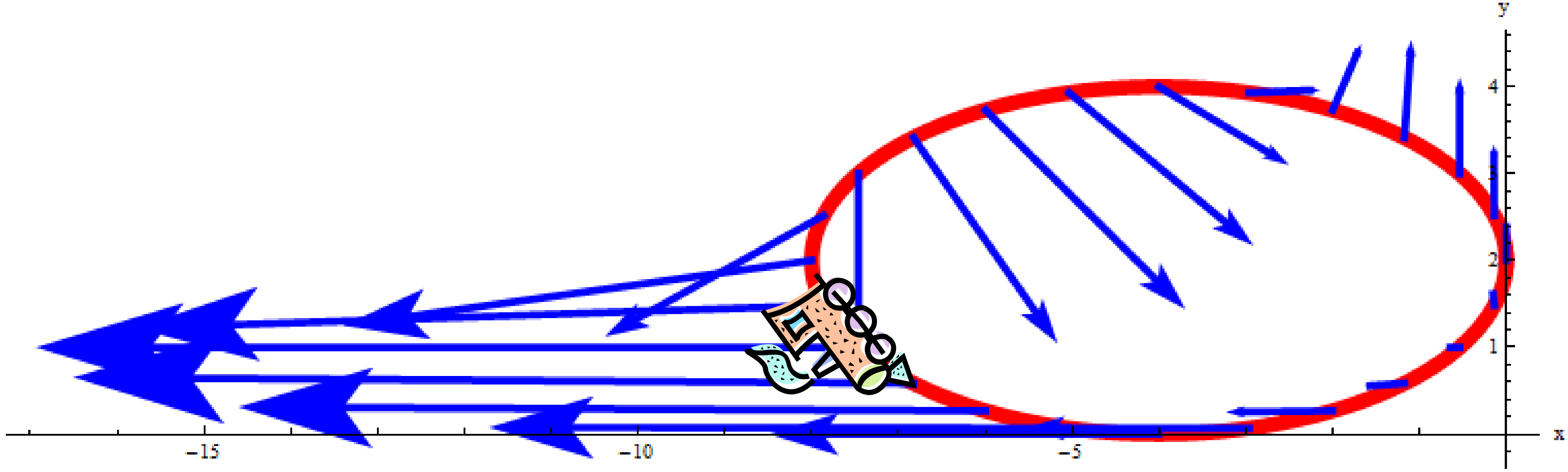


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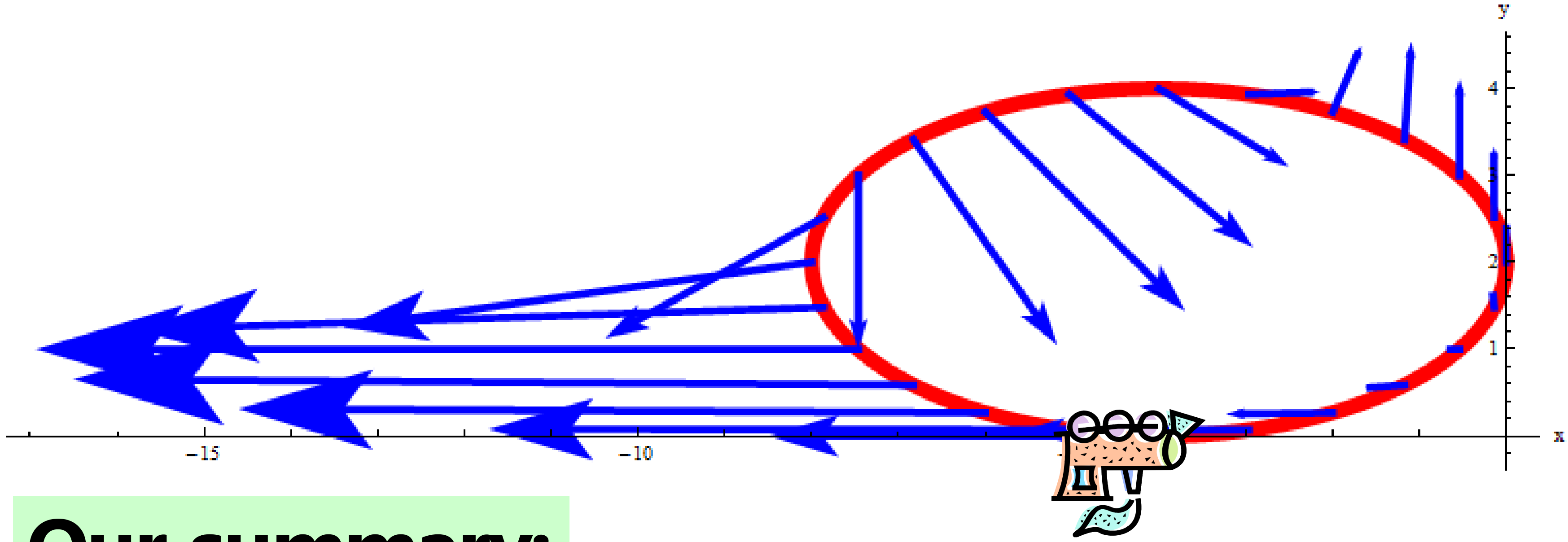
## Describe what the train experiences here:

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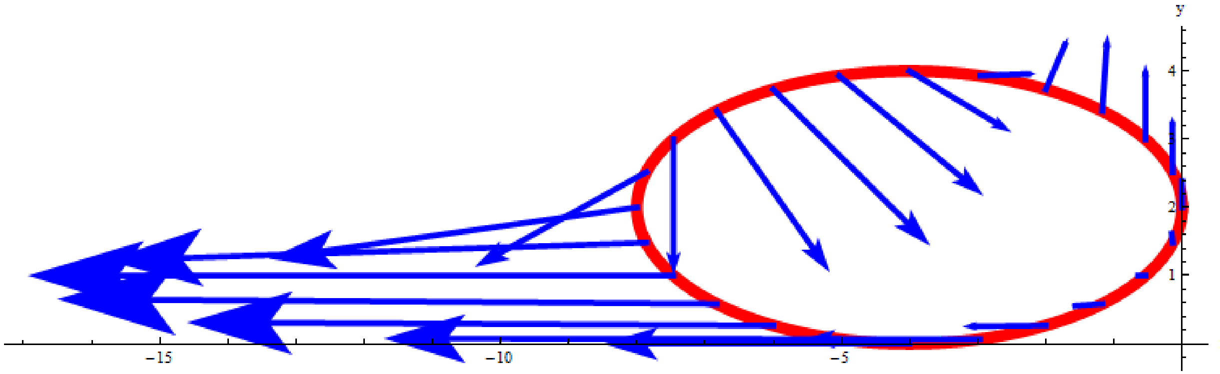
**Example 6**: Let the vector field $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$ act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$.

The language we used to describe what was happening to the train is actually very helpful for analyzing this scenario. Notice that we used words like forward or backward, left or right all RELATIVE to the train at its position on the track.




You might not have realized it, but you were breaking down the field vector down into two components: the component in the direction of the tangent vector to the curve, $(x'(t),y'(t))$ (forward/backward), and the component in the direction of the normal vector to the curve, $(y'(t),-x'(t))$ (left/right).

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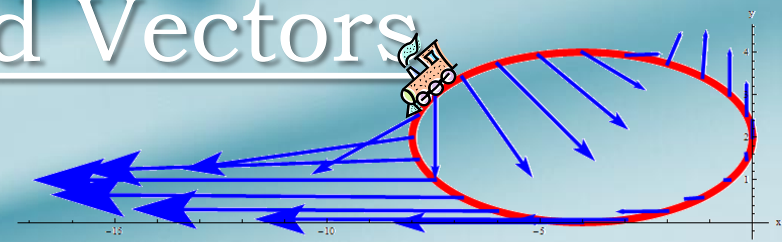
# Breaking Down the Field Vectors into Two Components



Let $\text{Field}(x,y) = (m(x,y), n(x,y))$ be a vector field acting on the curve $(x(t),y(t))$.

## Forward/Backward Push:

The push of the field vector in the direction of the tangent vector to the curve:

$$\left(\frac{\text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))}{(x'(t),y'(t)) \bullet (x'(t),y'(t))}\right)(x'(t),y'(t))$$

"The flow of the vector field ALONG the curve."

## Left/Right Push:

The push of the field vector in the direction of the normal vector to the curve:

$$\left(\frac{\text{Field}(x(t),y(t)) \bullet (y'(t),-x'(t))}{(y'(t),-x'(t)) \bullet (y'(t),-x'(t))}\right)(y'(t),-x'(t))$$

"The flow of the vector field ACROSS the curve."

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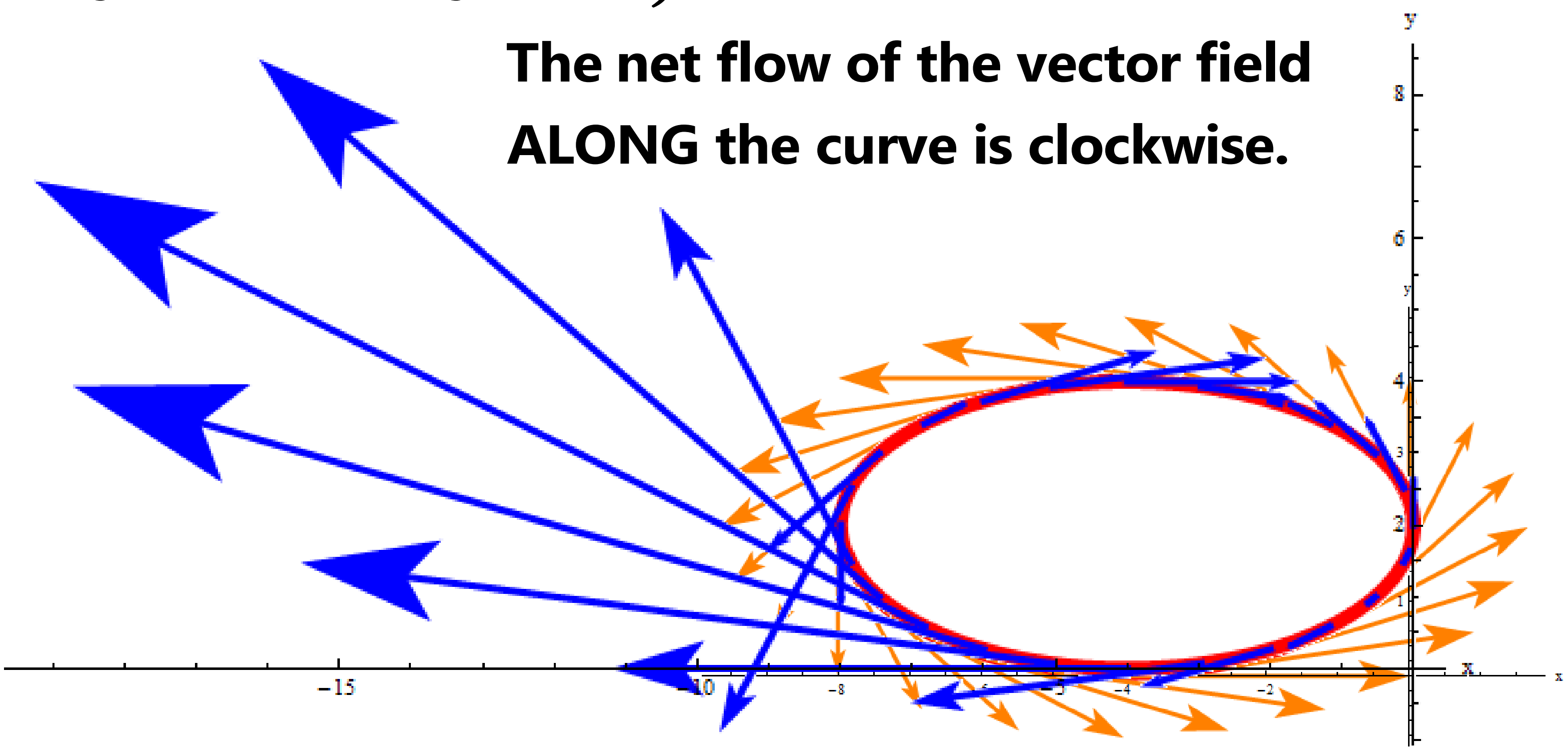
**Example 6 :** Let the vector field $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$ act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$.

Use the push of the field vectors in the direction of the tangent vectors to the curve to determine whether the net flow ALONG the curve is clockwise or counterclockwise:

$$\text{Plot}\left(\frac{\text{Field}(x(t),y(t))\bullet(x'(t),y'(t))}{(x'(t),y'(t))\bullet(x'(t),y'(t))}\right)(x'(t),y'(t))$$

The net flow of the vector field ALONG the curve is clockwise.

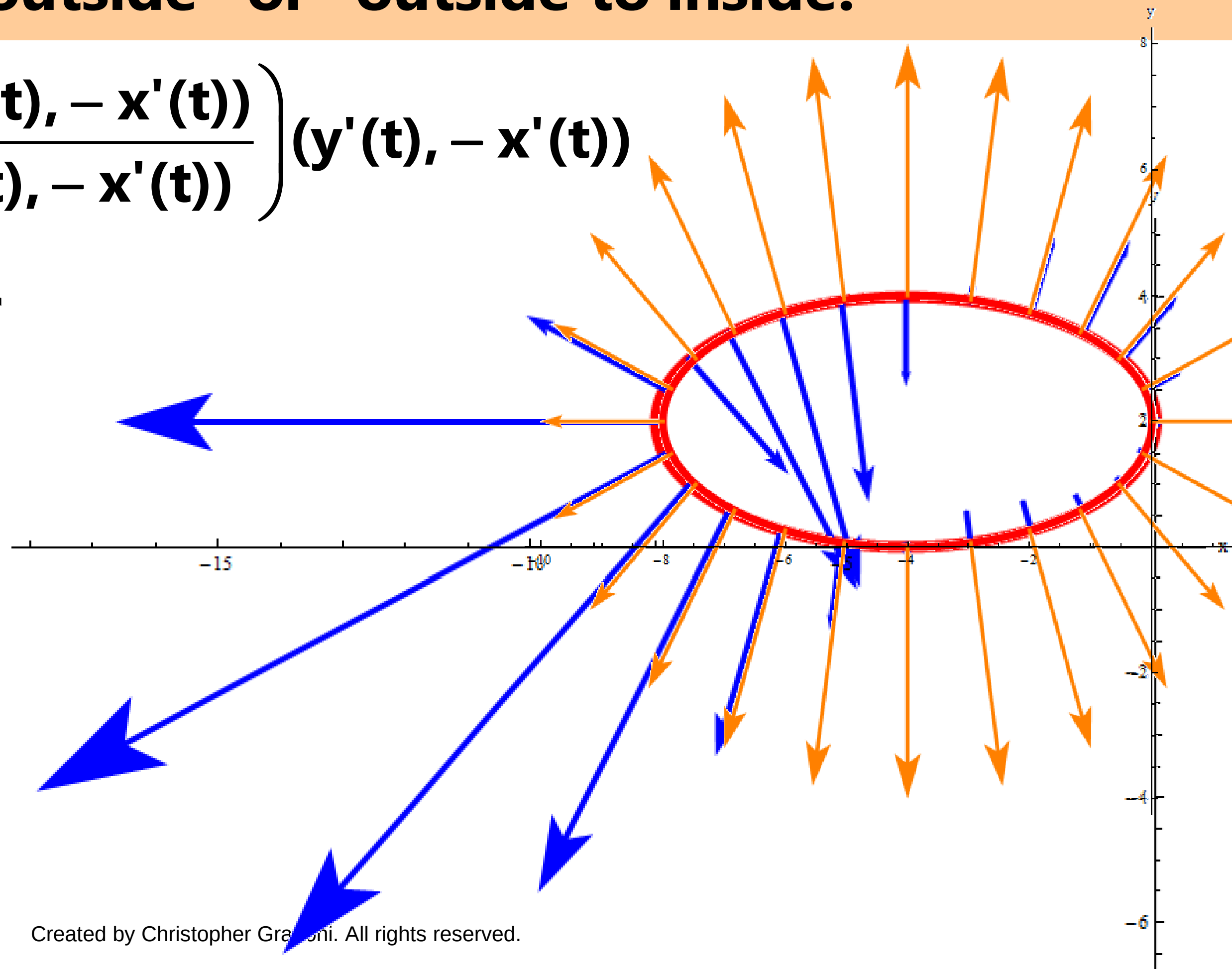


**Example 6:** Let the vector field $\left((3x^2-3x)(y-3),(3y^2-3y)(x+3)\right)$ act on the ellipse $(x(t),y(t))=(-4,2)+(4\cos(t),2\sin(t))$.

**Use the push of the field vectors in the direction of the normal vectors to the curve to determine whether the net flow ACROSS the curve is "inside to outside" or "outside to inside."**

$$\text{Plot}\left(\frac{\text{Field}(x(t),y(t))\bullet(y'(t),-x'(t))}{(y'(t),-x'(t))\bullet(y'(t),-x'(t))}\right)(y'(t),-x'(t))$$

**The net flow of the vector field ACROSS the curve is "inside to outside."**

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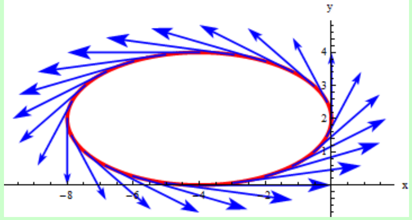
# Flow Along and Flow Across

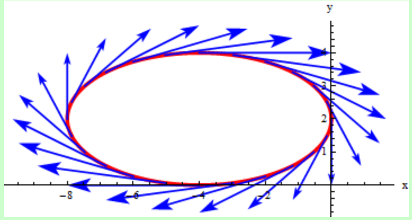
**"The flow of the vector field ALONG the curve."**

$$\text{Plot}\left(\frac{\text{Field}(x(t),y(t))\bullet(x'(t),y'(t))}{(x'(t),y'(t))\bullet(x'(t),y'(t))}\right)(x'(t),y'(t))$$

**Counterclockwise:**



**Clockwise:**

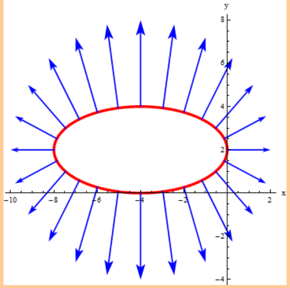


***Could be 0!**

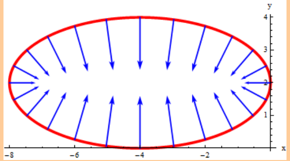
**"The flow of the vector field ACROSS the curve."**

$$\text{Plot}\left(\frac{\text{Field}(x(t),y(t))\bullet(y'(t),-x'(t))}{(y'(t),-x'(t))\bullet(y'(t),-x'(t))}\right)(y'(t),-x'(t))$$

**Inside to outside:**



**Outside to Inside:**



***Could be 0!**

**For this chapter, our only measurement tool is to plot and then eyeball the results...**

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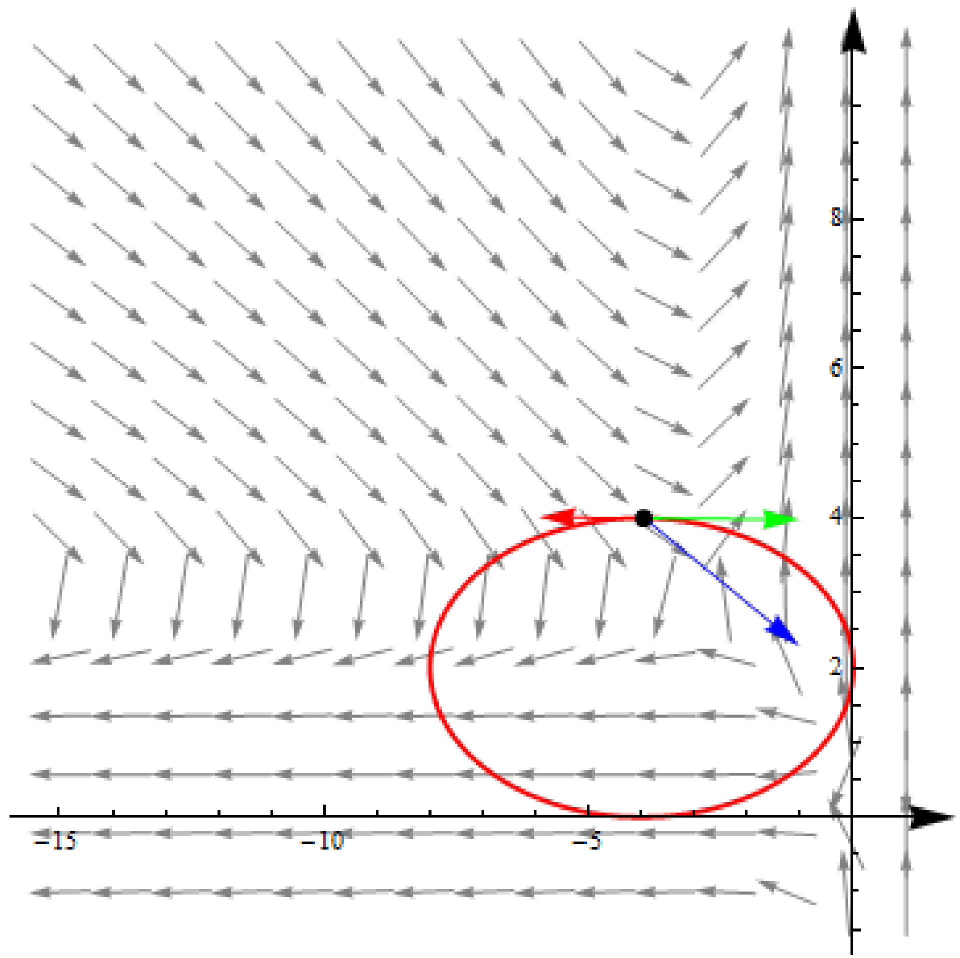
# Try the Animated Version for Flow ALONG the Curve

In the following demo, the black point travels along the red curve. The vector field acts on the point as it travels. We have a red unit tangent vector that shows us the direction of travel, while the blue vector shows the field vector whose tail lies on the curve at that point. Finally, the green vector is the projection of the blue field vector onto the red tangent vector.




This code is based on the following Wolfram Demonstration:

[http://demonstrations.wolfram.com/VectorFieldActingOnACurve/](http://demonstrations.wolfram.com/VectorFieldActingOnACurve/)

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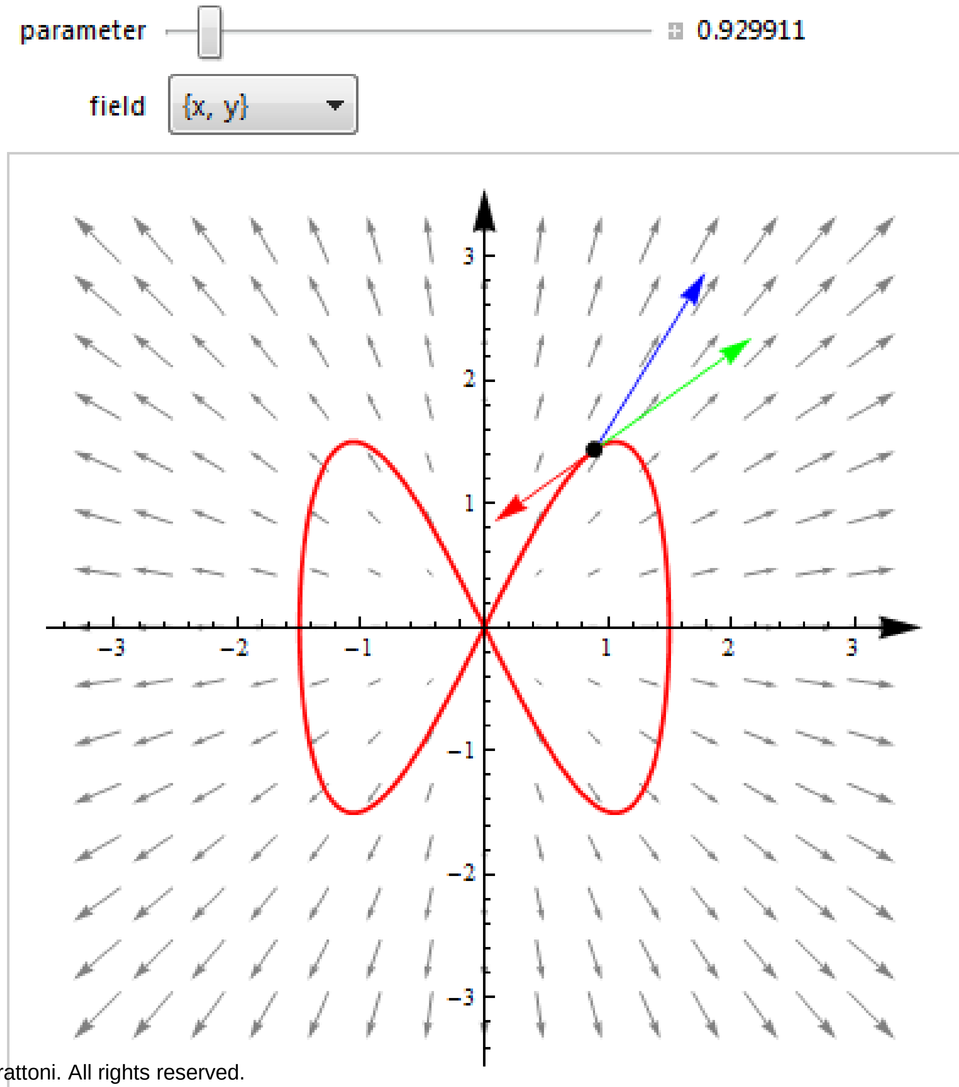
# Example 7: Try One Curve With Lots of Vector Fields to Choose From…

In the following demo, the black point travels along the red curve. The vector field acts on the point as it travels. We have a red unit tangent vector that shows us the direction of travel, while the blue vector shows the field vector whose tail lies on the curve at that point. Finally, the green vector is the projection of the blue field vector onto the red tangent vector.

This code is directly from the following Wolfram Demonstration:

[http://demonstrations.wolfram.com/VectorFieldActingOnACurve/](http://demonstrations.wolfram.com/VectorFieldActingOnACurve/)

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