Lesson 7

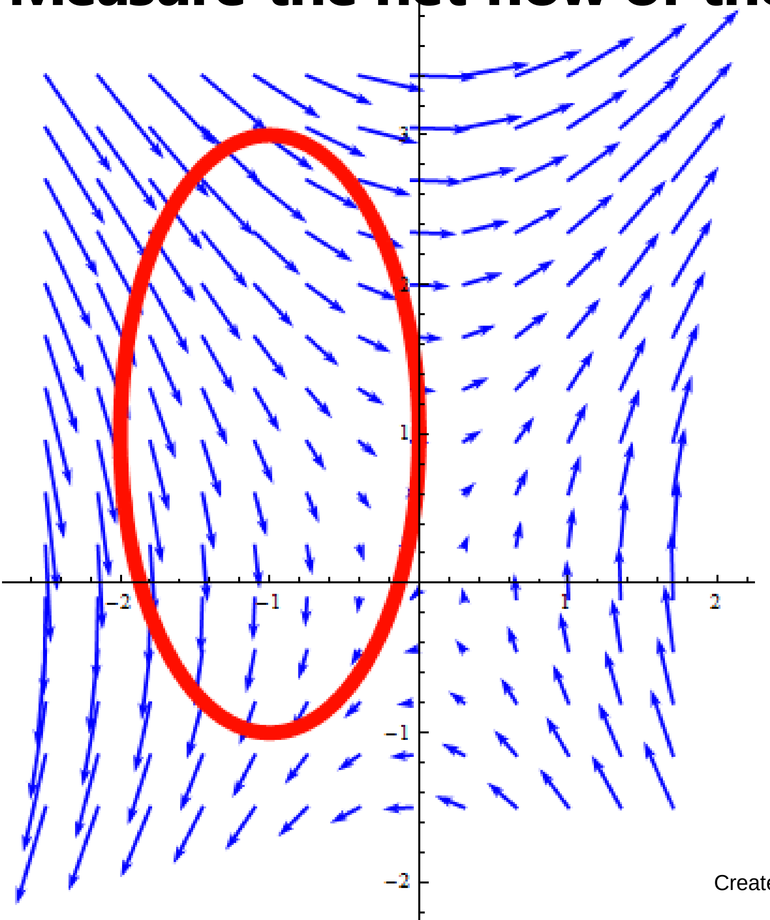
Integrals for Measuring Flow

Example 1: Measuring the Net Flow of a Vector Field ALONG a Closed Curve

Let $\text{Field}(x, y) = (y, 2x)$ be a vector field acting on the curve

$$E(t) = (x(t), y(t)) = (\cos(t), 2\sin(t)) + (-1, 1)$$

Measure the net flow of the vector field along the curve:



Recall that this is like a set of underwater train tracks being buffeted by a swirling, violent current.

We can get a better picture of what is going on by just plotting the field vectors whose tails are on the curve:

Example 1: Measuring the Net Flow of a Vector Field ALONG a Closed Curve

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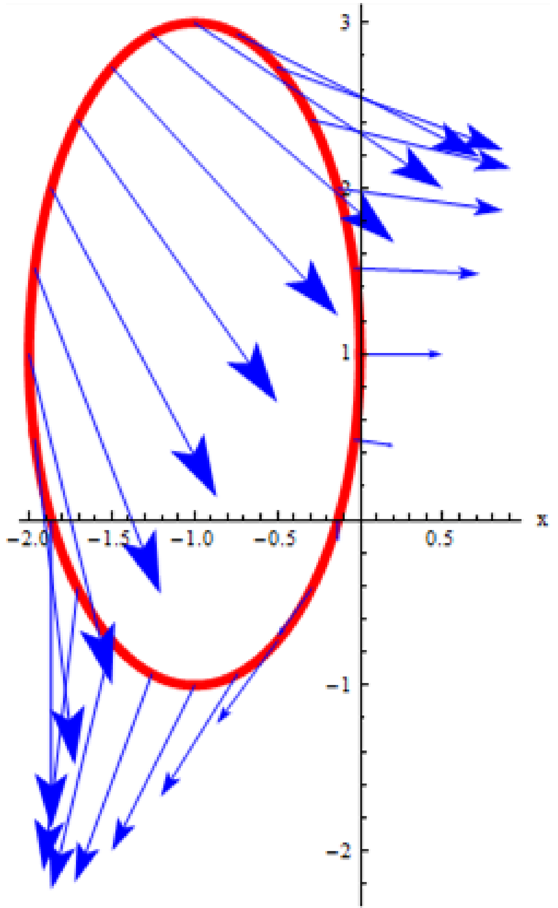

Can we tell if it is clockwise or counterclockwise yet?

How do we get a better picture of what is going on based on what we learned last chapter?

Example 1: Measuring the Net Flow of a Vector Field ALONG a Closed Curve

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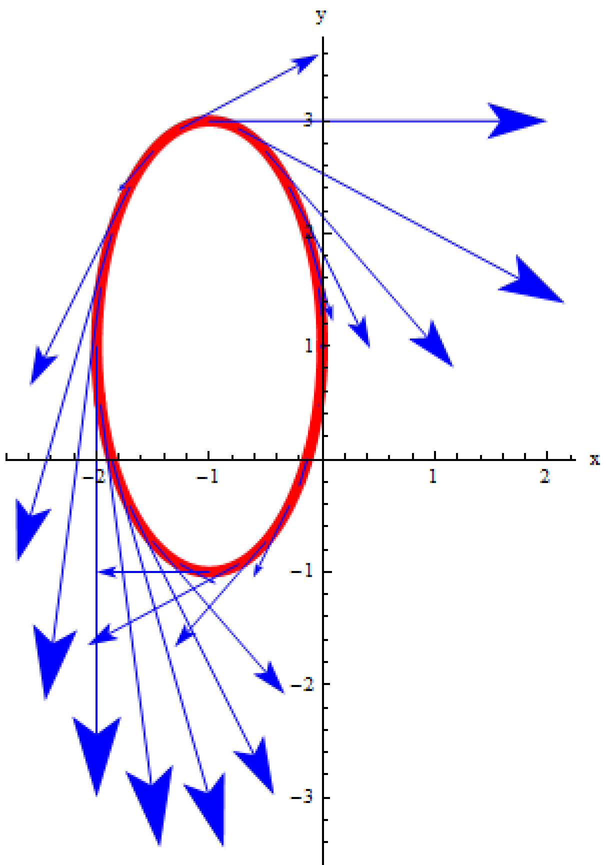

Yes, at each point, we can plot the push of the field vector in the direction of the tangent vector to the curve:

$$\frac{\text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))}{(x'(t), y'(t)) \bullet (x'(t), y'(t))}(x'(t), y'(t))$$

It looks like it is a net counterclockwise flow of the vector field along the curve, but are we 100% positive about this?

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Let $\text{Field}(x,y) = (y, 2x)$ be a vector field acting on the curve

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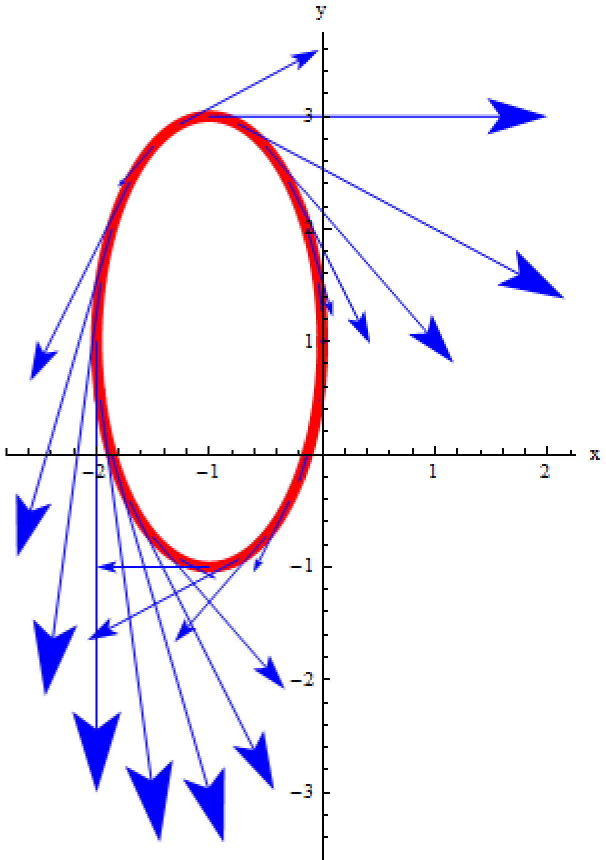

Our real measure is really what happens to $\text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))$ around the curve since this records when the field helps or hurts our movement in the direction of the parameterization, and by how much:

Integrate it :

$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))\, dt$$

Example 1: Measuring the Net Flow of a Vector Field ALONG a Closed Curve

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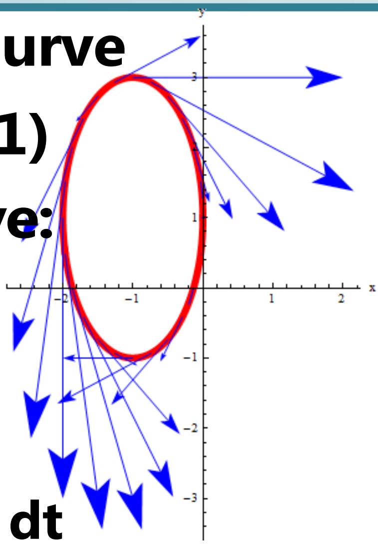


$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))\, dt$$

$$= \int_0^{2\pi} (2\sin(t) + 1, 2\cos(t) - 2) \bullet (-\sin(t), 2\cos(t))\, dt$$

$$= \int_0^{2\pi} -2\sin^2(t) - \sin(t) + 4\cos^2(t) - 4\cos(t)\, dt$$

$$= 2\pi$$

Positive! The net flow of the vector field along the curve is in the direction of the parameterization (counterclockwise).

The Line Integral: Formalizing What We Just Did

So far, we only integrated because we had a vague notion that integrating this expression will accumulate all of the little pushes with or against the curve and tell us whether we have a net push in the clockwise or counterclockwise direction. We can do better:

$$\int_a^b f(x)\, dx \qquad\qquad \int_C F(x,y) \bullet dc$$

○ **Function of a single variable**	○ **Two-component vector field, F(x,y)**
○ **Interval [a,b] along the x-axis**	○ **Curve in space, C, with parameterization c**
○ **Integrate with respect to left-right (x) movement: dx**	○ **Integrate with respect to movement along the parameterization of the curve: dc**
○ **f(x) "times" dx**	○ **F(x, y) "dot product" dc**

The Line Integral: Formalizing What We Just Did

Let C be a curve parameterized by $c(t)=(x(t),y(t))$ for $a \le t \le b$, and let F(x,y) be a vector field, $F(x,y)=\text{Field}(x,y)=\left(m(x,y),n(x,y)\right)$.

$$\int_C F(x,y)\bullet dc = \int_a^b F(x(t),y(t))\bullet \frac{dc}{dt}dt$$

$$= \int_a^b F(x(t),y(t))\bullet c'(t)dt$$

$$= \int_a^b \text{Field}(x(t),y(t))\bullet (x'(t),y'(t))dt$$

This is called the line (path) integral of the vector field along the curve.

Other Ways Of Writing The Line Integral

Let $\text{Field}(x,y) = \left(m(x,y), n(x,y)\right)$ and let C be a curve parameterized by (x(t),y(t)) for $a \le t \le b$.

$$\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt$$

$$= \int_a^b \left(m(x(t), y(t)), n(x(t), y(t))\right) \bullet (x'(t), y'(t))dt$$

$$= \int_a^b \left(m(x(t), y(t))x'(t) + n(x(t), y(t))y'(t)\right)dt$$

$$= \int_a^b m(x(t), y(t))\frac{dx}{dt}dt + n(x(t), y(t))\frac{dy}{dt}dt$$

$$= \int_C m(x,y)dx + n(x,y)dy$$

Put it All Together: Measuring the Net Flow of a Vector Field ALONG a Curve

If C is a closed curve with a counterclockwise parameterization:	If C is not closed:
Integral : $\int_a^b \text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))dt$ $= \int_a^b \big(m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t)\big)dt$ $= \oint_C m(x,y)dx + n(x,y)dy$	**Integral :** $\int_a^b \text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))dt$ $= \int_a^b \big(m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t)\big)dt$ $= \int_C m(x,y)dx + n(x,y)dy$
These integrals are used to compute net flow of the vector field **along** the closed curve (**clockwise or counterwise or 0**). They can be modified to measure flow of the vector field across the closed curve (inside to outside or outside to inside) with relative ease.	Without a closed curve, the integral is measured whether the net flow **along** the open curve is in the direction of parameterization or against it, and whether the net flow across the open curve is from "above to below" or "below to above."

The physics interpretation of "flow along" is "work."

Put it All Together: Measuring the Net Flow of a Vector Field ALONG a Curve

Let C be a closed curve with a COUNTERCLOCKWISE parameterization :

If $\oint_C m(x,y)dx + n(x,y)dy > 0$, then the net flow of the vector field along the curve is counterclockwise.

If $\oint_C m(x,y)dx + n(x,y)dy < 0$, then the net flow of the vector field along the curve is clockwise.

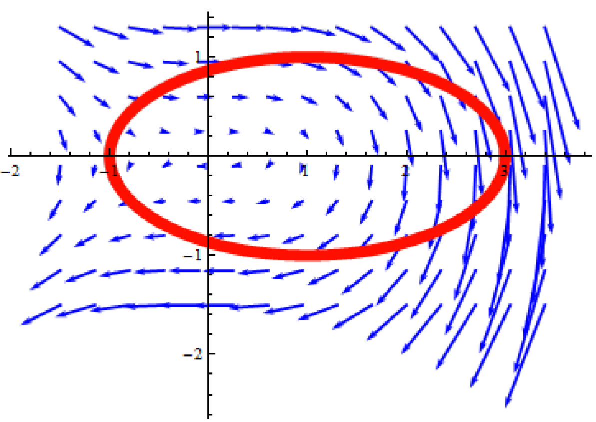
$\oint_C m(x,y)dx + n(x,y)dy$ can equal 0.

Example 2: Measuring the Net Flow of a Vector Field Along Another Closed Curve

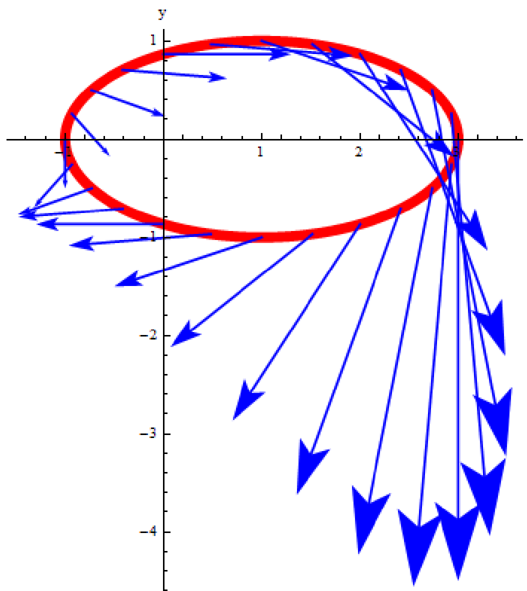
Let $\text{Field}(x,y) = (3y, -x^2)$ **be a vector field acting on the ellipse** $\left(\dfrac{x-1}{2}\right)^2 + y^2 = 1.$

Compute $\oint_C m(x,y)\,dx + n(x,y)\,dy$ **:**

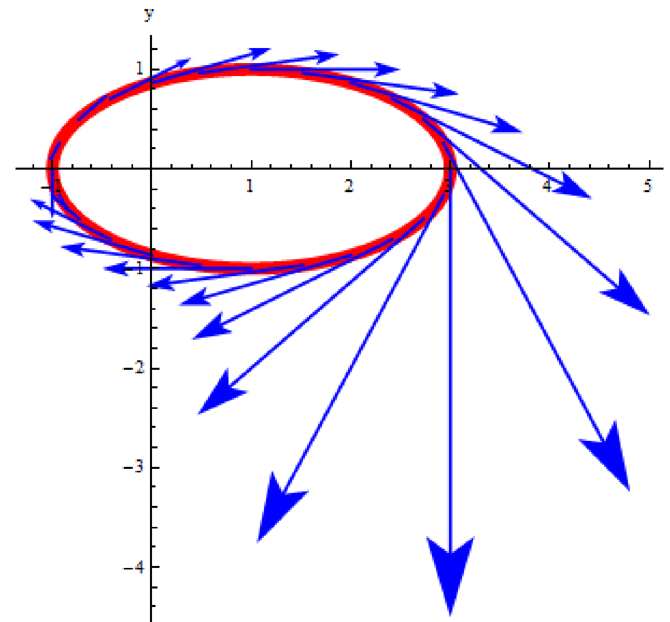
Field and Curve :



Vectors on Curve :



Component of Field Vectors in the Direction of the Tangent Vectors



Example 2: Measuring the Net Flow of a Vector Field Along Another Closed Curve

Let $\text{Field}(x,y) = (3y, -x^2)$ be a vector field acting on the ellipse $\left(\frac{x-1}{2}\right)^2 + y^2 = 1$. Compute $\oint_C m(x,y)dx + n(x,y)dy$:

Counterclockwise

$$\text{Field}(x,y) = \big(m(x,y), n(x,y)\big) = (3y, -x^2)$$

$$E(t) = (x(t), y(t)) = (2\cos(t), \sin(t)) + (1,0)$$

$$\oint_C m(x,y)dx + n(x,y)dy = \int_a^b \big(m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t)\big)dt$$

$$= \int_0^{2\pi} \cos(t) - 4\cos^2(t) - 4\cos^3(t) - 6\sin^2(t)dt$$

$$= -10\pi$$

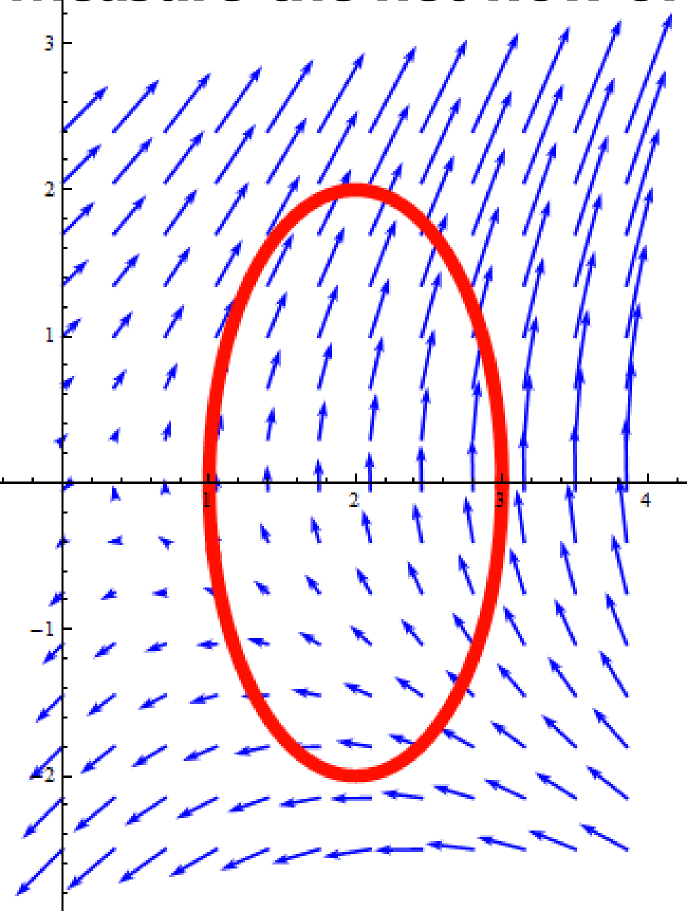
Negative! The net flow of the vector field along the curve is against the direction of the parameterization (clockwise).

Example 3: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve

Let $\text{Field}(x, y) = (y, x + y)$ **be a vector field acting on the curve**

$$E(t) = (x(t), y(t)) = (\cos(t), 2\sin(t)) + (2, 0)$$

Measure the net flow of the vector field across the curve:



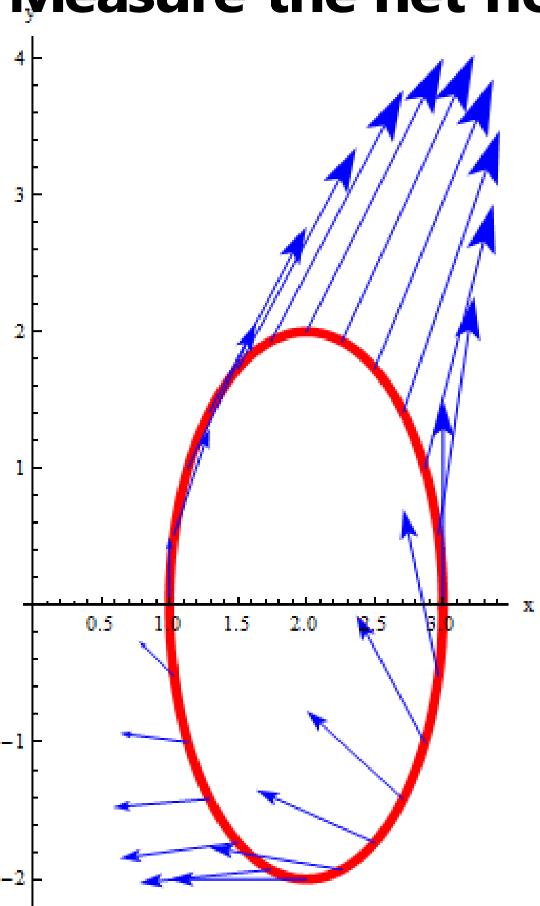
We can get a better picture of what is going on by just plotting the field vectors whose tails are on the curve:

Example 3: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve

Let $\text{Field}(x,y)=(y,x+y)$ **be a vector field acting on the curve**

$$E(t)=(x(t),y(t))=(\cos(t),2\sin(t))+(2,0)$$

Measure the net flow of the vector field across the curve:



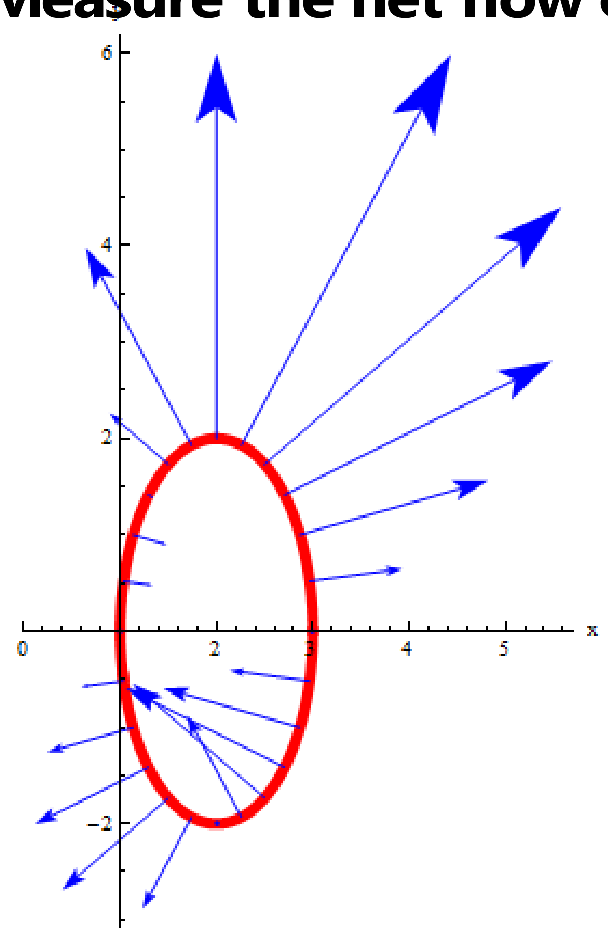
Better, but we can improve our picture further:

Example 3: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve

Let $\text{Field}(x,y) = (y, x+y)$ be a vector field acting on the curve

$$E(t) = (x(t), y(t)) = (\cos(t), 2\sin(t)) + (2,0)$$

Measure the net flow of the vector field across the curve:



At each point, we can plot the push of the field vector in the direction of the normal vector to the curve:

$$\frac{\text{Field}(x(t),y(t)) \bullet (y'(t), -x'(t))}{(y'(t),-x'(t)) \bullet (y'(t),-x'(t))}(y'(t),-x'(t))$$

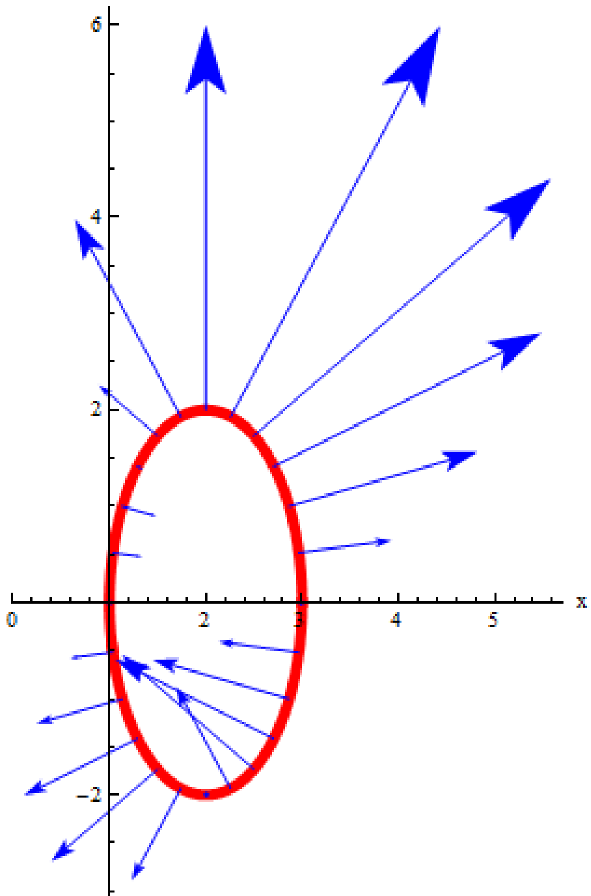
It looks like it is a net flow of the vector field across the curve from inside to outside, but we want to verify this numerically:

Example 3: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve

Let $\text{Field}(x,y) = (y, x+y)$ be a vector field acting on the curve

$$E(t) = (x(t), y(t)) = (\cos(t), 2\sin(t)) + (2,0)$$

Measure the net flow of the vector field across the curve:

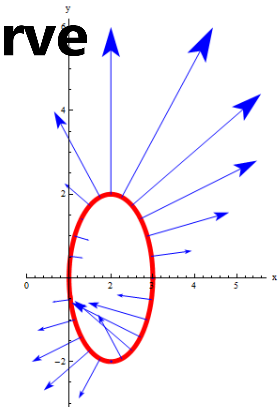


Our real measure is really what happens to $\text{Field}(x(t), y(t)) \bullet (y'(t), -x'(t))$ around the curve since this records when the field pushes us left or right relative to our direction of travel, and by how much:

Integrate it :

$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (y'(t), -x'(t))\, dt$$

Example 3: Measuring the Net Flow of a Vector Field ACROSS a Closed Curve



Let $\text{Field}(x,y) = (y, x+y)$ be a vector field acting on the curve

$$E(t) = (x(t), y(t)) = (\cos(t), 2\sin(t)) + (2,0)$$

Measure the flow of the vector field across the curve:

$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (y'(t), -x'(t))\, dt$$

$$= \int_0^{2\pi} (2\sin(t), \cos(t) + 2\sin(t) + 2) \bullet (2\cos(t), \sin(t))\, dt$$

$$= \int_0^{2\pi} 2\sin(t) + 5\cos(t)\sin(t) + 2\sin^2(t)\, dt$$

$$= 2\pi$$

Positive! The net flow of the vector field across the curve is with the direction of the normal vectors. For a counterclockwise parameterization, this is from inside to outside.

Put it All Together: Measuring the Net Flow of a Vector Field ACROSS a Curve

If C is a closed curve with a counterclockwise parameterization:	**If C is not closed:**
Integral : $$\int_a^b \text{Field}(x(t),y(t)) \bullet (y'(t),-x'(t))dt$$ $$= \int_a^b \left(-n(x(t),y(t))x'(t) + m(x(t),y(t))y'(t)\right)dt$$ $$= \oint_C -n(x,y)dx + m(x,y)dy$$	**Integral :** $$\int_a^b \text{Field}(x(t),y(t)) \bullet (y'(t),-x'(t))dt$$ $$= \int_a^b \left(-n(x(t),y(t))x'(t) + m(x(t),y(t))y'(t)\right)dt$$ $$= \int_C -n(x,y)dx + m(x,y)dy$$
With a closed curve, the integral measures whether the net flow ***ACROSS*** the closed curve is from "inside to outside" or from "outside to inside."	Without a closed curve, the integral measures whether the net flow ***ACROSS*** the open curve is from above to below the curve or from below to above.

The physics interpretation of "flow across" is "flux."

Put it All Together: Measuring the Net Flow of a Vector Field ACROSS a Curve

Let C be a closed curve with a COUNTERCLOCKWISE parameterization :

If $\oint_C -n(x,y)dx + m(x,y)dy > 0$, then the net flow (flux) of the vector field across the curve is from inside to outside.

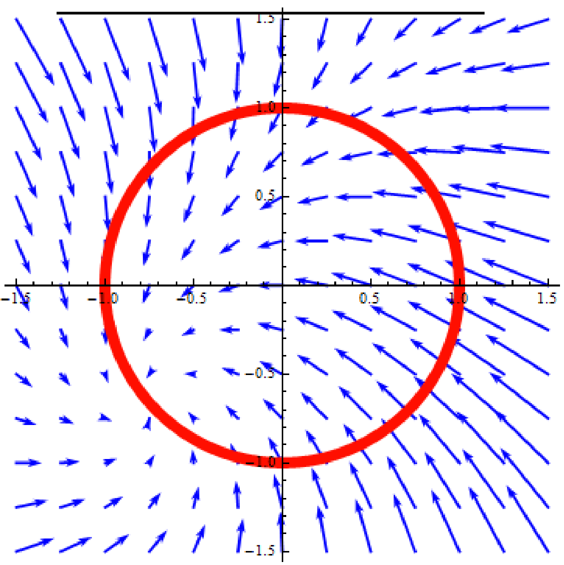
If $\oint_C -n(x,y)dx + m(x,y)dy < 0$, then the net flow (flux) of the vector field across the curve is from outside to inside.

$\oint_C -n(x,y)dx + m(x,y)dy$ can equal 0.

Example 4: Measuring the Net Flow of a Vector Field Across Another Closed Curve

Let $\text{Field}(x,y) = (-x-\cos(y), -y+\sin(x))$ be a vector field acting on the circle $x^2+y^2=1$. Compute $\oint_C -n(x,y)\,dx + m(x,y)\,dy$:

Field and Curve :



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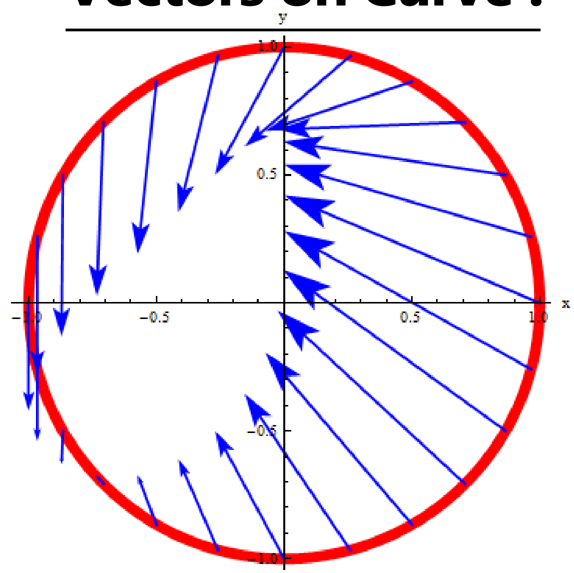
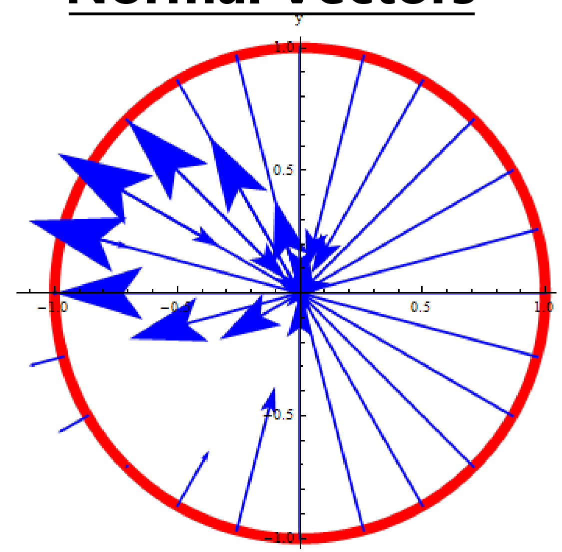


Component of Field Vectors in the Direction of the Normal Vectors



Example 4: Measuring the Net Flow of a Vector Field Across Another Closed Curve

Let $\text{Field}(x,y) = (-x - \cos(y), -y + \sin(x))$ be a vector field acting on the circle $x^2 + y^2 = 1$. Compute $\oint_C -n(x,y)dx + m(x,y)dy$:

Counterclockwise

$$\text{Field}(x,y) = \big(m(x,y), n(x,y)\big)$$
$$= (-x - \cos(y), -y + \sin(x))$$

$$c(t) = (x(t), y(t))$$
$$= (\cos(t), \sin(t))$$

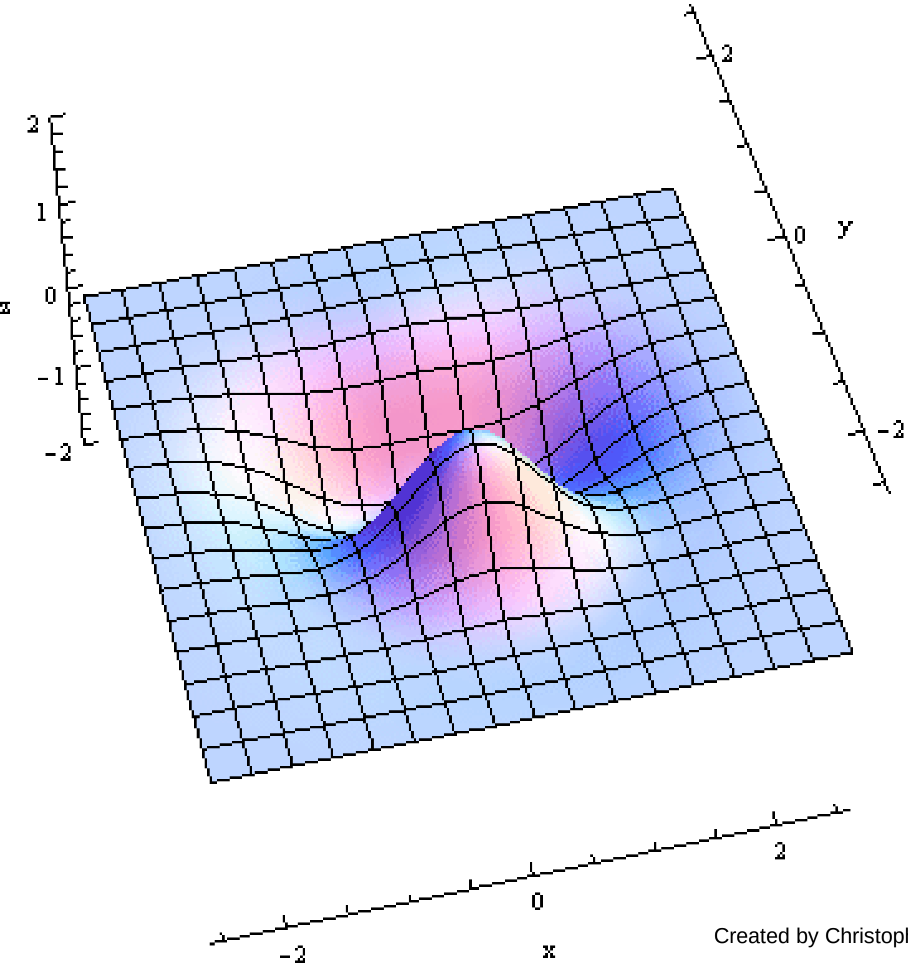
$$\oint_C -n(x,y)dx + m(x,y)dy = \int_a^b \big(-n(x(t),y(t))x'(t) + m(x(t),y(t))y'(t)\big)dt$$

$$= -2\pi$$

Negative! The net flow of the vector field across the curve is from outside to inside.

Example 5: The Net Flow of a Gradient Field Along a Closed Curve

Start with a surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

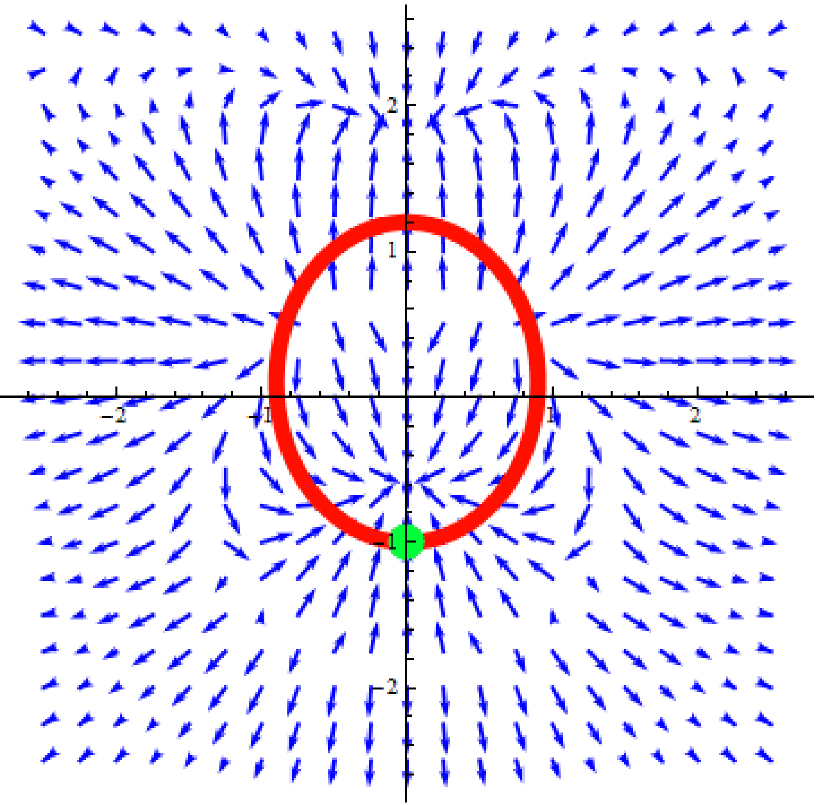


Now let's take a look at the gradient field, ∇f, associated with the surface:

Example 5: The Net Flow of a Gradient Field Along a Closed Curve

Now plot $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ **for** $f(x,y) = -\frac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$ **:**

Consider an elliptical path (x(t),y(t)) on the surface for $0 \le t \le 2\pi$**:**



$$\left(0.9\cos\left(t+\frac{3\pi}{2}\right), 1.1\sin\left(t+\frac{3\pi}{2}\right)\right) + (0, 0.1)$$

Compute the net flow of the gradient field along the curve:

$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt$$

$$= \int_0^{2\pi} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet (x'(t), y'(t))dt$$

$$= \int_0^{2\pi} \frac{df(x(t), y(t))}{dt} dt \quad \text{(Chain Rule)}$$

Let's interpret this using the surface...

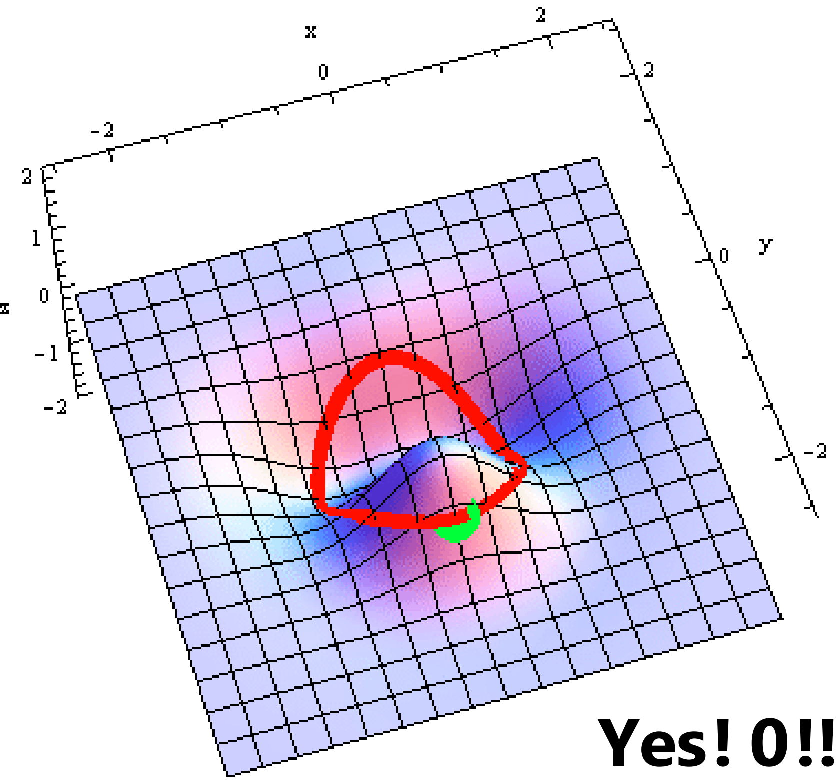
Example 5: The Net Flow of a Gradient Field Along a Closed Curve

$$\int_0^{2\pi} \text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))dt = \int_0^{2\pi} \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right) \bullet (x'(t),y'(t))dt$$

$$= \int_0^{2\pi} \frac{df(x(t),y(t))}{dt}dt$$

$$= f(x(2\pi),y(2\pi)) - f(x(0),y(0))$$

$$= 0$$



This represents the net change in height (altitude) on the curve from 0 to 2π along the path (x(t),y(t)).

What should this be for our closed curve?

Yes! 0!! The net flow of a gradient field along a closed curve is 0.

Summary: The Net Flow of a Gradient Field Along a Closed Curve

Let Field(x,y)=$\left(m(x,y), n(x,y)\right)$ be a gradient field, and let C be a simple closed curve with a parameterization (x(t),y(t)) for $a \le t \le b$.

1) $\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt = 0$

2) $\oint_C m(x,y)dx + n(x,y)dy = 0$

3) **The net flow of a gradient field along a simple closed curve is 0.**

Why is this intuitively true?

How do we know a closed curve can't be a trajectory of a gradient field?

Is the net flow of a gradient field ACROSS a closed curve 0?

Summary: The Net Flow of a Gradient Field Along a Closed Curve

Let Field(x,y) = $\left(m(x,y), n(x,y)\right)$ be a gradient field, and let C be a simple closed curve with a parameterization (x(t),y(t)) for $a \le t \le b$.

1) $\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt = 0$

2) $\oint_C m(x,y)dx + n(x,y)dy = 0$

3) The net flow of a gradient field along a simple closed curve is 0.

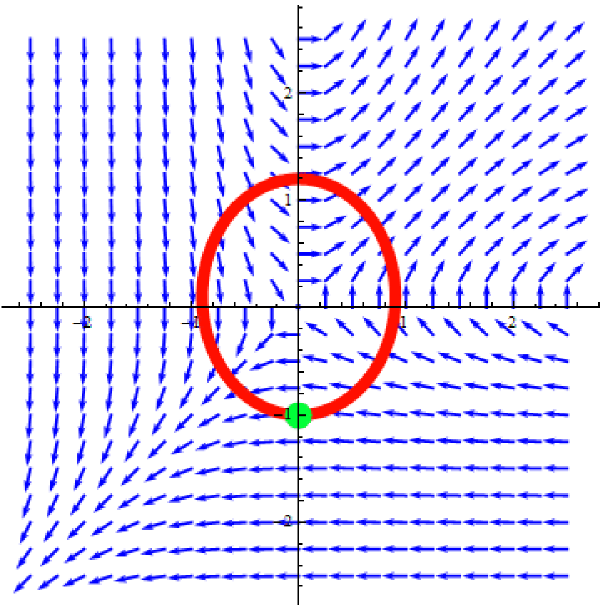
For this reason, gradient fields are called <u>conservative vector fields</u>. We also say that gradient fields are <u>irrotational</u>.

Example 6: Is the Net Flow of ANY Vector Field Along a Closed Curve Zero?

Let $\text{Field}(x,y) = \left(ye^{x}, xe^{y}\right)$ and keep the same curve, 👍 *Counterclockwise*

$$\left(0.9\cos\left(t+\frac{3\pi}{2}\right), 1.1\sin\left(t+\frac{3\pi}{2}\right)\right) + (0, 0.1)$$

$$\int_0^{2\pi} \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt \approx 0.548155$$



So the net flow of a general vector field along a closed curve need not equal 0.

The net flow of the vector field along the curve is counterclockwise.

Example 7: How Do We Know When We Have a Gradient Field?

The last two examples clearly show the advantages to knowing whether our vector field is a gradient field or not. This motivates us to ask the question, "Given a vector field, how do we know if it is a gradient field?"

Here are a few vector fields. Try to classify them as "gradient fields", or "not gradient fields":

Vector Field	Gradient?	Why?
$\left(xy^2, x^2y\right)$		
$\left(ye^x, xe^y\right)$		
$\left(x+1, -y+4\right)$		

The Gradient Test

A vector field, Field(x,y) = $\left(m(x,y), n(x,y)\right)$, is a gradient field if and only if it is defined at all points (x,y) and:

$$\frac{\partial n(x,y)}{\partial x} = \frac{\partial m(x,y)}{\partial y}$$

Proof of "if" Part of Theorem:

If Field(x,y) is a gradient field, then Field(x,y) = $\left(f_x, f_y\right)$.

So $f_{xy} = f_{yx}$.

Gradient Test (Summary):

The Gradient Test :

1) $$\frac{\partial n}{\partial x} = \frac{\partial m}{\partial y}$$

2) No singularities

Example 7: How Do We Know When We Have a Gradient Field?

Try again to classify these vector fields as "gradient fields", or "not gradient fields":

Vector Field	Gradient?	Why?
$\left(xy^2, x^2y\right)$		
$\left(ye^x, xe^y\right)$		
$\left(x+1, -y+4\right)$		

Example 7: How Do We Know When We Have a Gradient Field?

Find a function f(x,y) that yields $\nabla f(x,y) = \left(xy^2 + 2, x^2y - 1 \right)$.

We want an f(x,y) such that $\frac{\partial f}{\partial x} = xy^2 + 2$ **and** $\frac{\partial f}{\partial y} = x^2y - 1$:

If $\frac{\partial f}{\partial x} = xy^2 + 2$, **then** $f(x,y) = \int (xy^2 + 2)dx = \frac{x^2y^2}{2} + 2x + \phi(y)$.

Hence, $\frac{\partial f}{\partial y} = x^2y + \phi'(y)$.

If $\frac{\partial f}{\partial y} = x^2y + \phi'(y)$ **and** $\frac{\partial f}{\partial y} = x^2y - 1$, **then** $\phi'(y) = -1$.

So $\phi(y) = -y + c \Rightarrow f(x,y) = \frac{x^2y^2}{2} + 2x - y + c$

Example 7: How Do We Know When We Have a Gradient Field? (ALTERNATE SOLUTION)

Find a function f(x,y) that yields $\nabla f(x,y) = \left(xy^2 + 2, x^2y - 1 \right)$.

We want an f(x,y) such that $\frac{\partial f}{\partial x} = xy^2 + 2$ and $\frac{\partial f}{\partial y} = x^2y - 1$:

If $\frac{\partial f}{\partial y} = x^2y - 1$, then $f(x,y) = \int (x^2y - 1)\,dy = \frac{x^2y^2}{2} - y + \psi(x)$.

Hence, $\frac{\partial f}{\partial x} = xy^2 + \psi'(x)$.

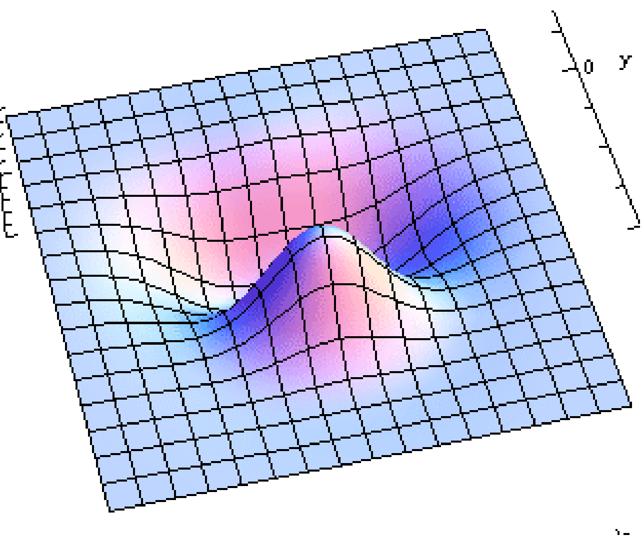
If $\frac{\partial f}{\partial x} = xy^2 + \psi'(x)$ and $\frac{\partial f}{\partial x} = xy^2 + 2$, then $\psi'(x) = 2$.

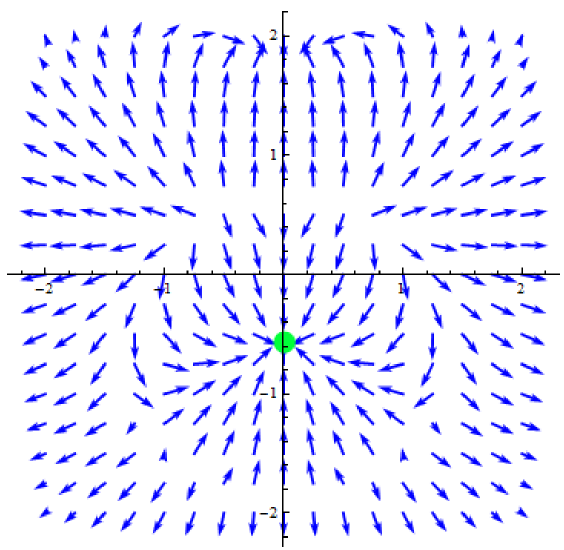
So $\psi(x) = 2x + c \Rightarrow f(x,y) = \frac{x^2y^2}{2} - y + 2x + c$

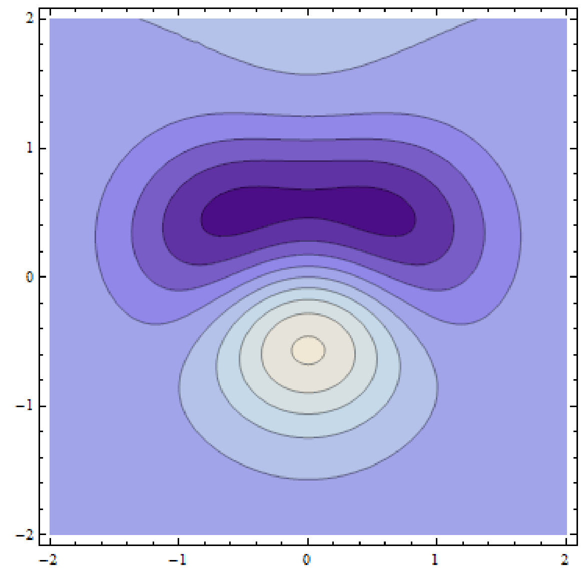
Example 8: The Net Flow of a Gradient Field Along an Open Curve (Path Independence)

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

Let's look at the surface, the gradient field, and a contour plot:



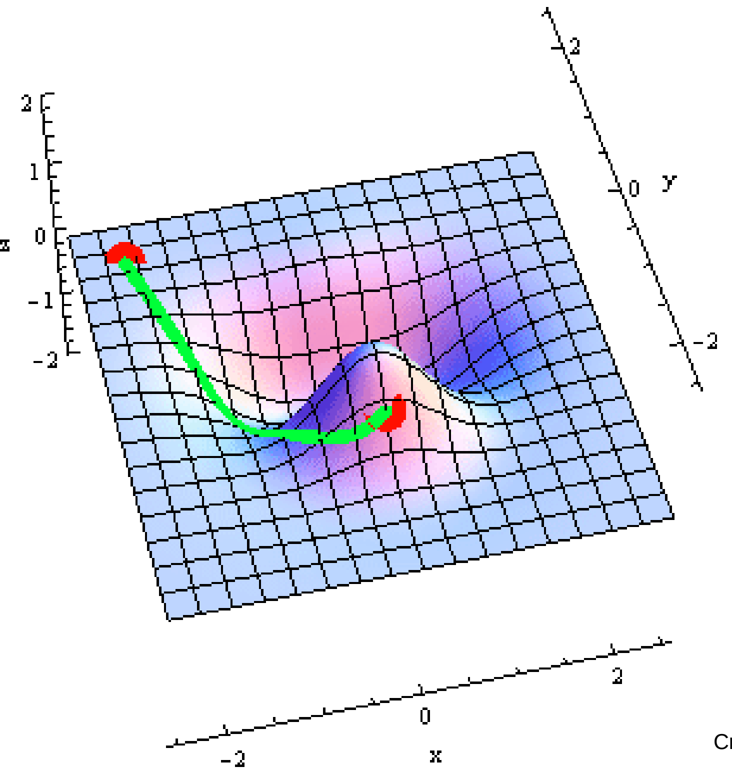




Example 8: The Net Flow of a Gradient Field Along an Open Curve (Path Independence)

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

Let's compute $\int_C m(x,y)dx + n(x,y)dy$ **for** C_1, C_2, **and** C_3 **:**



$C_1 : (x_1(t), y_1(t)) = \left(t, t^2 + \dfrac{t}{2} - 1\right) \quad -2 \le t \le 0$

$$\int_{-2}^{0} \text{Field}(x_1(t), y_1(t)) \bullet (x_1'(t), y_1'(t))dt \approx 0.597$$

$$\int_{-2}^{0} \frac{df(x_1(t), y_1(t))}{dt} dt \approx 0.597 \qquad \text{(Chain Rule)}$$

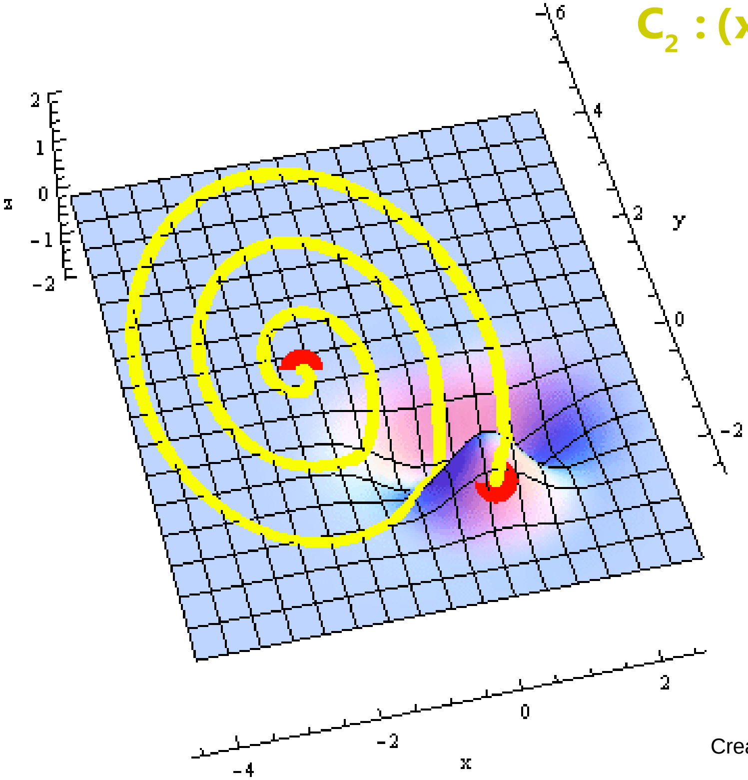
Our change in height from the starting point to end point is up 0.597 units.

Example 8: The Net Flow of a Gradient Field Along an Open Curve (Path Independence)

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x)+3y\cos(y)}{e^{x^2+y^2}}$

What might you expect $\int_C m(x,y)dx + n(x,y)dy$ **for** C_2 **to equal?**

$$C_2 : (x_2(t), y_2(t)) = \left(\frac{16t\cos(t)}{25\pi\sqrt{2}} - 2, -\frac{24t\sin(t)}{25\pi\sqrt{2}} + 2\right) \quad 0 \le t \le \frac{25\pi}{4}$$



$$\int_0^{25\pi/4} \text{Field}(x_2(t), y_2(t)) \bullet (x_2'(t), y_2'(t))dt \approx 0.597$$

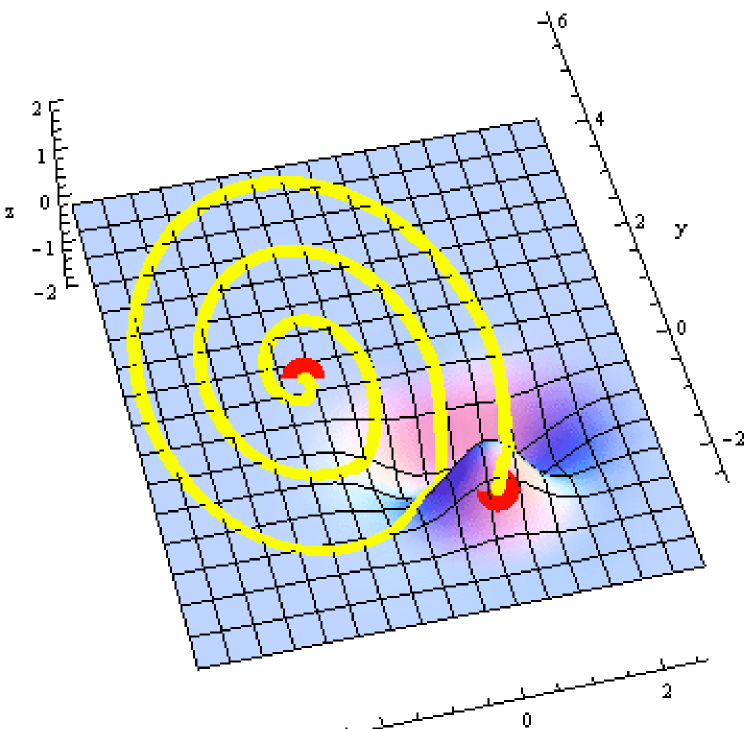
$$\int_0^{25\pi/4} \frac{df(x_2(t), y_2(t))}{dt} dt \approx 0.597 \qquad \text{(Chain Rule)}$$

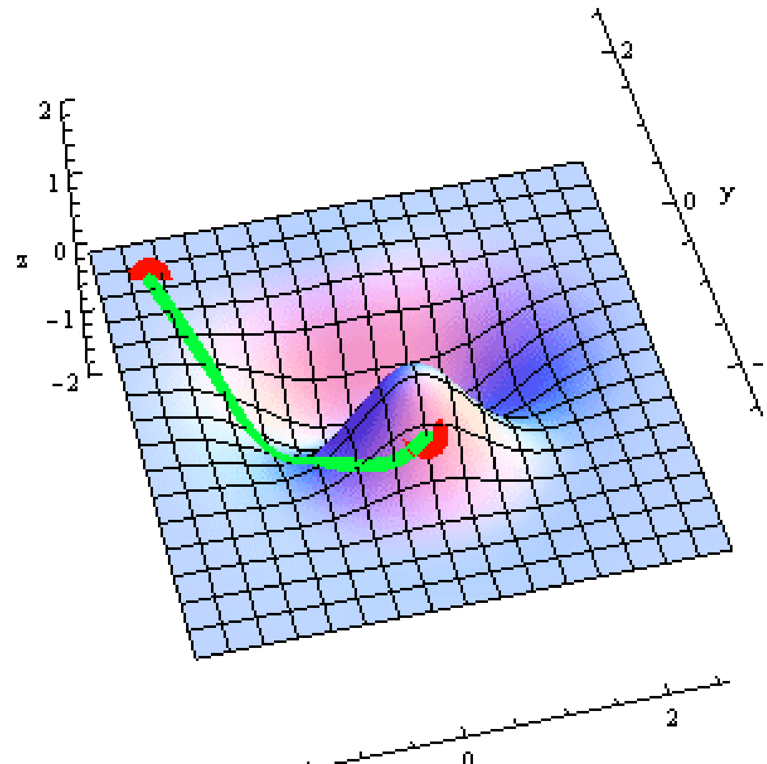
Our change in height from the starting point to end point is up 0.597 units.

Example 8: The Net Flow of a Gradient Field Along an Open Curve (Path Independence)

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

Why did that happen? Can we generalize this phenomenon?

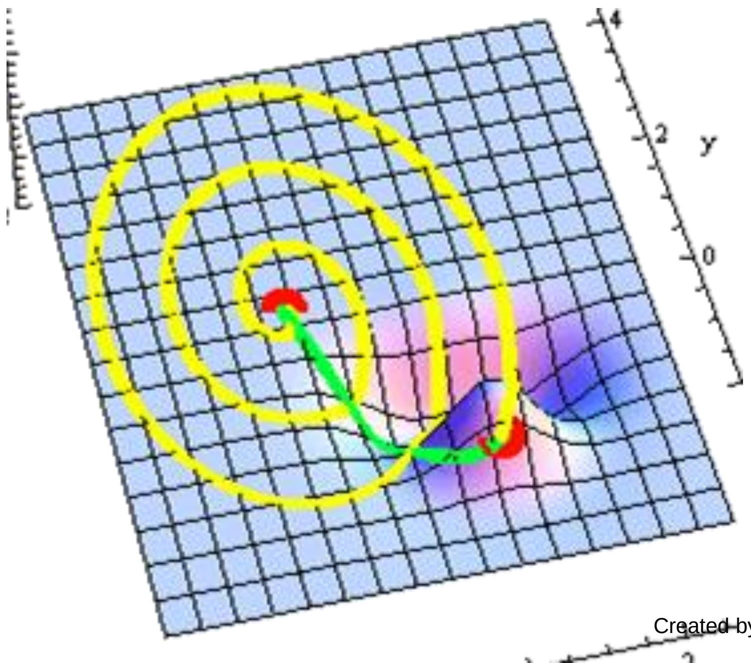




Path Independence: The Net Flow of a Gradient Field Along an Open Curve

Let $\text{Field}(x,y)=\big(m(x,y),n(x,y)\big)$ be a gradient field, and let C_1 and C_2 be different curves that share the same starting and ending point:

$$\int_{C_1} m(x,y)dx+n(x,y)dy=\int_{C_2} m(x,y)dx+n(x,y)dy$$

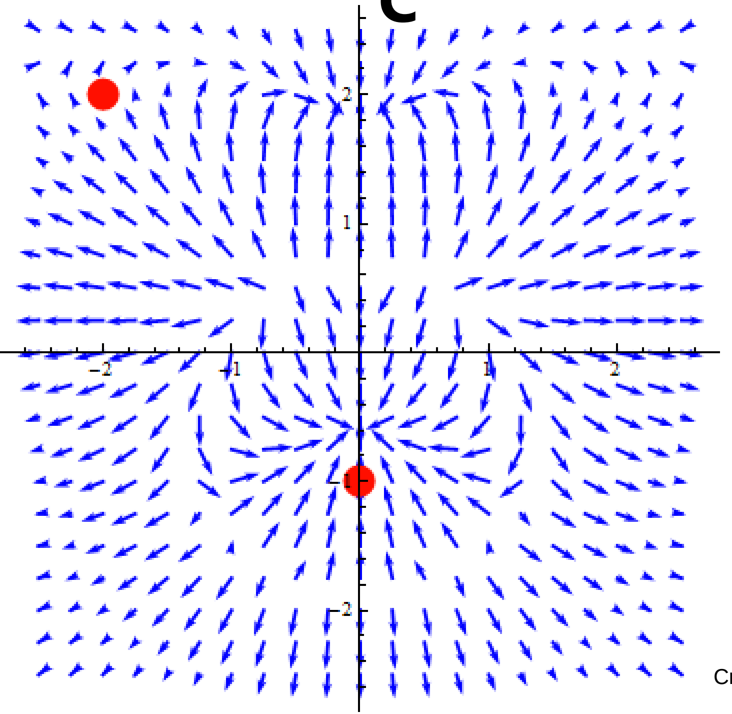


A gradient field is said to be path independent. The net flow of a gradient field along any two curves connecting the same two points is the same…

Example 8: The Net Flow of a Gradient Field Along an Open Curve

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

So what is the simplest path from (-2,2) to (0,-1) so we can find $\int_C m(x,y)dx + n(x,y)dy$ **??**



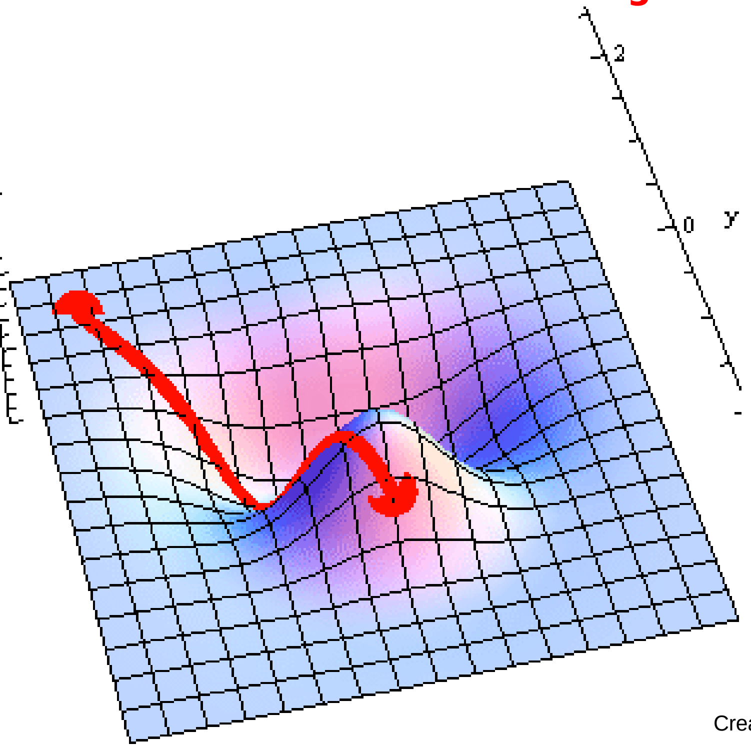
Hint: What's the most direct path between two points??

Example 8: The Net Flow of a Gradient Field Along an Open Curve

Start with a familiar surface $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

That's right! Let C_3 be a line connecting (-2,2) and (0,-1):

$$C_3 : (x_3(t), y_3(t)) = (-2,2) + t(2,-3) \qquad 0 \le t \le 1$$



$$\int_0^1 \text{Field}(x_3(t), y_3(t)) \bullet (x_3'(t), y_3'(t))dt$$

$$= \int_0^1 \frac{df(x_3(t), y_3(t))}{dt} dt \approx 0.597 \qquad \text{(Chain Rule)}$$

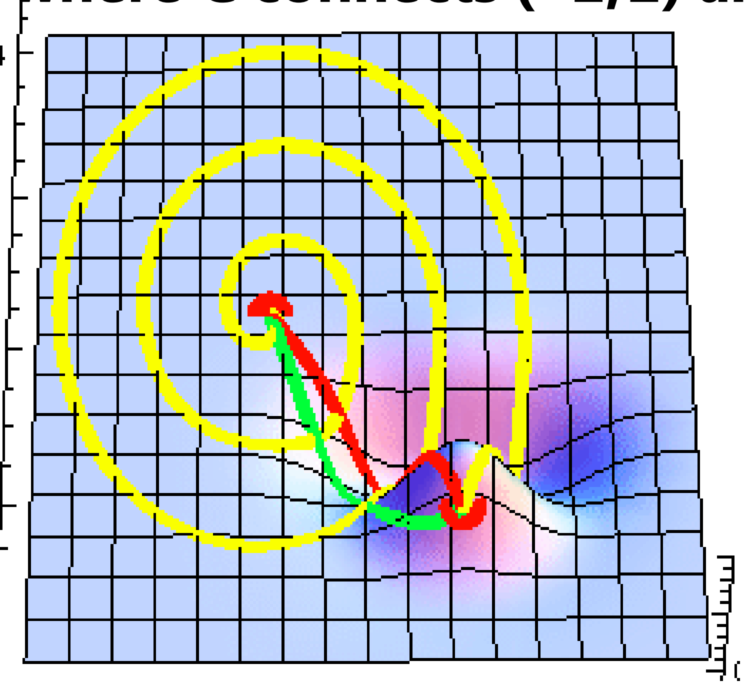
Our change in height from the starting point to end point is up 0.597 units.

Using "path independence" in this way is called "picking a replacement curve."

Example 8: The Net Flow of a Gradient Field Along an Open Curve

Find $\int_C m(x,y)dx + n(x,y)dy$ for $f(x,y) = -\dfrac{2x\sin(x) + 3y\cos(y)}{e^{x^2+y^2}}$

where C connects $(-2,2)$ and $(0,-1)$. What is an even easier way??



$$\int_a^b \text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))dt$$

$$= \int_a^b \frac{df(x(t),y(t))}{dt}dt$$

$$= f(x(b),y(b)) - f(x(a),y(a))$$

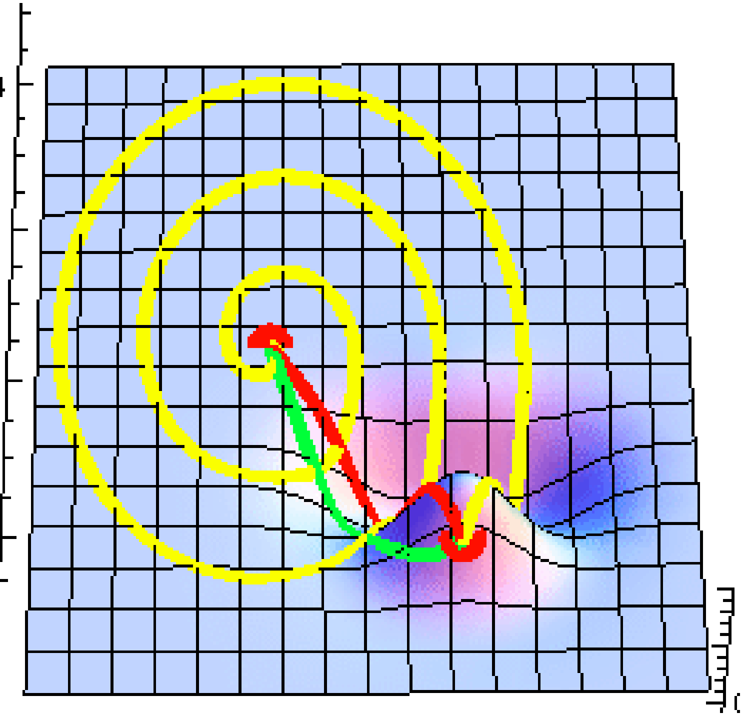
$$= f(0,-1) - f(-2,2)$$

$$\approx 0.597$$

If you know the equation for the surface, then your easiest bet is to just find your change in height! $f(x_2,y_2) - f(x_1,y_1)$!!

The Fundamental Theorem of Line Integrals

$$\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt = \int_a^b \frac{df(x(t), y(t))}{dt} dt$$



$$= f(x(b), y(b)) - f(x(a), y(a))$$

The net flow of a gradient field along a closed curve is just your change in height:

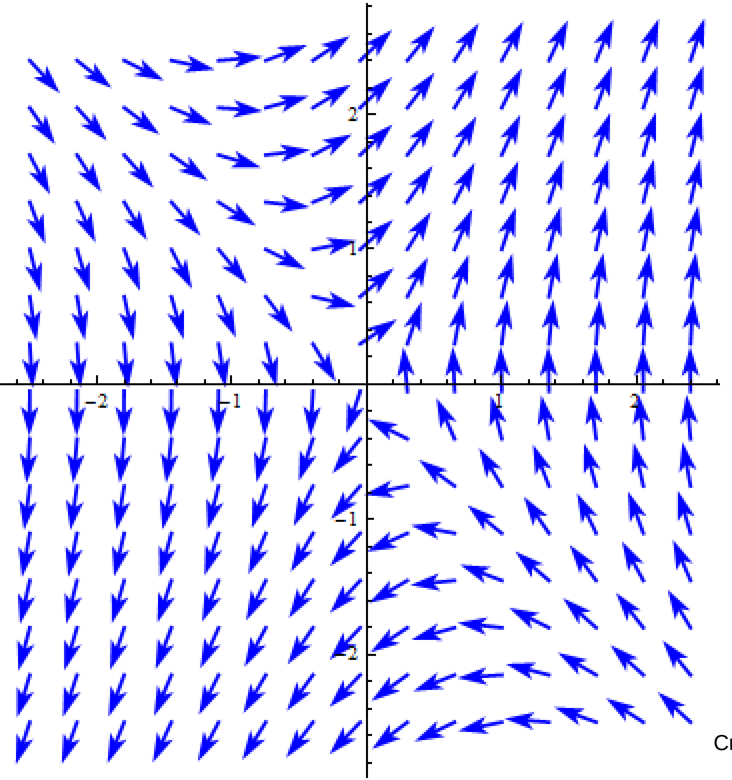
$$f(x_2, y_2) - f(x_1, y_1)$$

Fundamental Theorem of Line Integrals

Example 9: The Net Flow of a Vector Field Along an Open Curve

Start with a vector field $\left(m(x,y), n(x,y)\right) = \left(y, 2x+y\right)$.

a) Is our vector field a gradient field?



$$\frac{\partial m}{\partial y} = 1 \qquad \frac{\partial n}{\partial x} = 2$$

Not a gradient field!

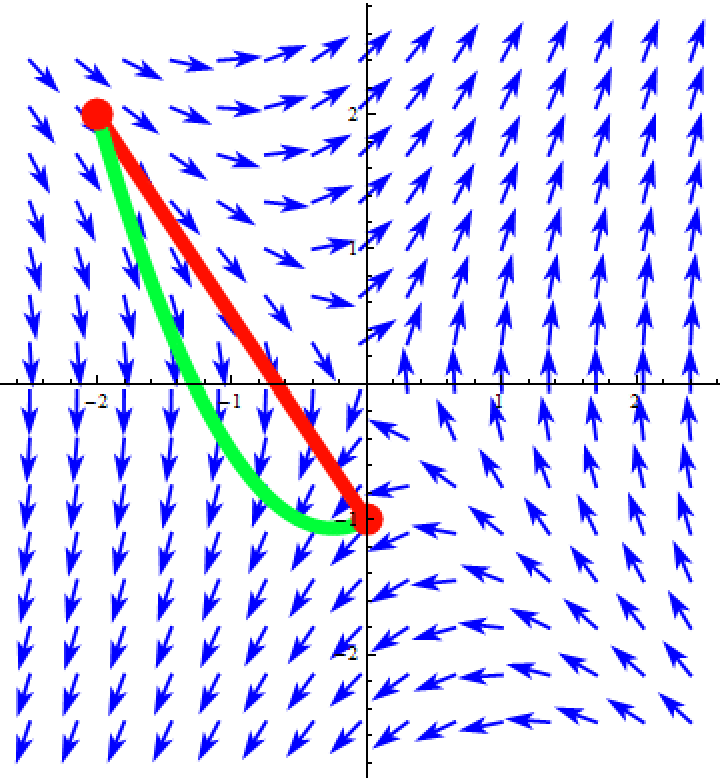
Example 9: The Net Flow of a Vector Field Along an Open Curve

Start with a vector field $\left(m(x,y),n(x,y)\right)=\left(y,2x+y\right)$.

b) Calculate $\int_C m(x,y)dx+n(x,y)dy$ **for** C_1 **and** C_2 **:**

$C_1:(x_1(t),y_1(t))=(-2,2)+t(2,-3) \quad 0\le t\le 1$

$C_2:(x_2(t),y_2(t))=\left(t,t^2+\frac{t}{2}-1\right) \quad -2\le t\le 0$



$$\int_0^1 \text{Field}(x_1(t),y_1(t))\bullet(x_1'(t),y_1'(t))dt$$

$$=\int_0^1 -9t+10dt$$

$$=\frac{11}{2}$$

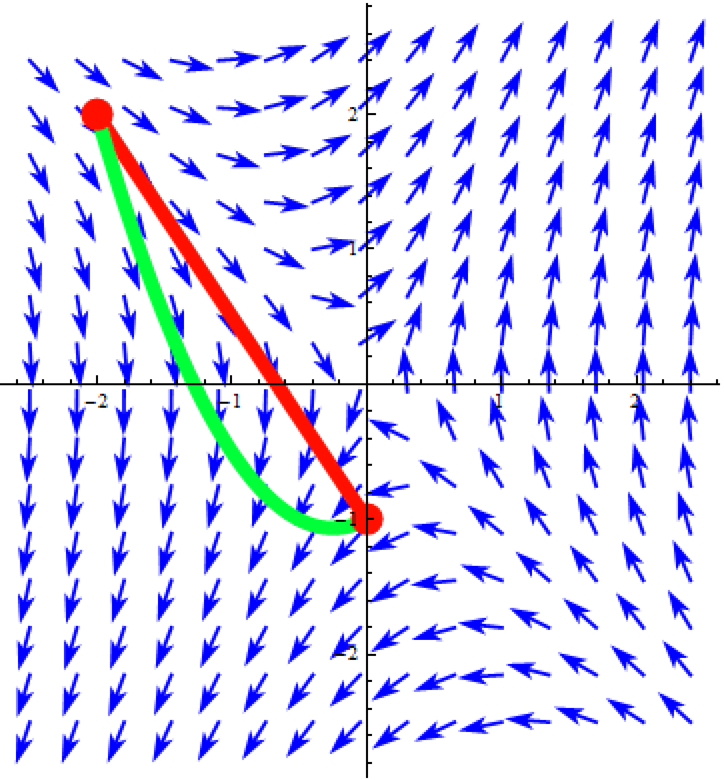
The net flow of the vector field along curve C_1 **is in the direction of parameterization.**

Example 9: The Net Flow of a Vector Field Along an Open Curve

Start with a vector field $\left(m(x,y),n(x,y)\right)=\left(y,2x+y\right)$.

b) Calculate $\int_C m(x,y)dx+n(x,y)dy$ **for** C_1 **and** C_2 **:**

$C_1:(x_1(t),y_1(t))=(-2,2)+t(2,-3) \quad 0\le t\le 1$



$C_2:(x_2(t),y_2(t))=\left(t,t^2+\frac{t}{2}-1\right) \quad -2\le t\le 0$

$$\int_{-2}^{0}\text{Field}(x_2(t),y_2(t))\bullet(x_2{}'(t),y_2{}'(t))dt$$

$$=\int_{-2}^{0}2t^3+\frac{13}{2}t^2-\frac{t}{4}-\frac{3}{2}dt$$

$$=\frac{41}{6}$$

The net flow of the vector field along curve C_2 is in the direction of parameterization.

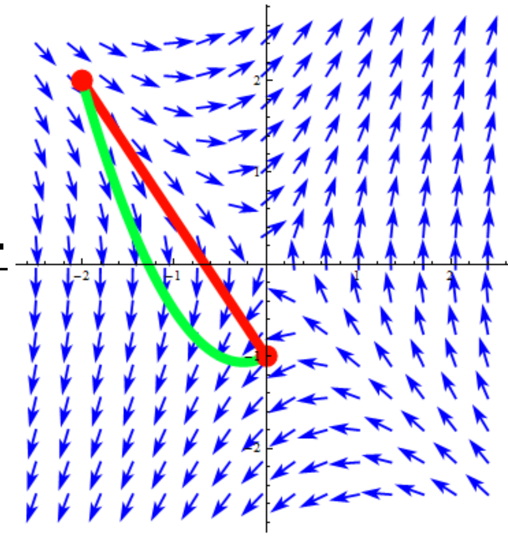
Example 9: Only Gradient Fields Are Path Independent?

Start with a non-gradient vector field Field(x,y)=$\left(y, 2x+y\right)$.

Calculate $\int_C m(x,y)dx + n(x,y)dy$ for C_1 and C_2 :

$$C_1 : (x_1(t), y_1(t)) = (-2,2) + t(2,-3) \quad 0 \le t \le 1$$

$$\int_0^1 \text{Field}(x_1(t), y_1(t)) \bullet (x_1'(t), y_1'(t))dt = \int_0^1 -9t + 10dt = \frac{11}{2}$$



$$C_2 : (x_2(t), y_2(t)) = \left(t, t^2 + \frac{t}{2} - 1\right) \quad -2 \le t \le 0$$

$$\int_{-2}^0 \text{Field}(x_2(t), y_2(t)) \bullet (x_2'(t), y_2'(t))dt = \int_{-2}^0 2t^3 + \frac{13}{2}t^2 - \frac{t}{4} - \frac{3}{2}dt = \frac{41}{6}$$

No path independence for a non-gradient vector field!!

Be Careful! Is it a Gradient Field?

We define a gradient field to be a vector field $(m(x,y),n(x,y))$ such that there exists a function $z=f(x,y)$ such that

$$(m(x,y),n(x,y)) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$$

Gradient fields have the property that there is a surface associated with them. Sources correspond with minima and sinks correspond with maxima.

WARNING: In order to be classified as a gradient field, your vector field must not have any points (x,y) where $(m(x,y),n(x,y))$ is undefined.

If your vector field is undefined at a point, it can't be called a gradient field. It doesn't enjoy all of the properties you usually get for free with a gradient field.

Example 11: Is Passing the First Part of the Gradient Test Enough?

Is $\left(\frac{1}{x^3y^2}, \frac{1}{x^2y^3} \right)$ a gradient field?

$$m(x,y) = \frac{1}{x^3y^2}, \text{ then } \frac{\partial m}{\partial y} = -\frac{2}{x^3y^3}$$

$$n(x,y) = \frac{1}{x^2y^3}, \text{ then } \frac{\partial n}{\partial x} = -\frac{2}{x^3y^3}$$

This looks like a gradient field, but technically it is not because we have singularities at all points such that $x = 0$ or $y = 0$.