Lesson 8

Sources, Sinks, and Singularities

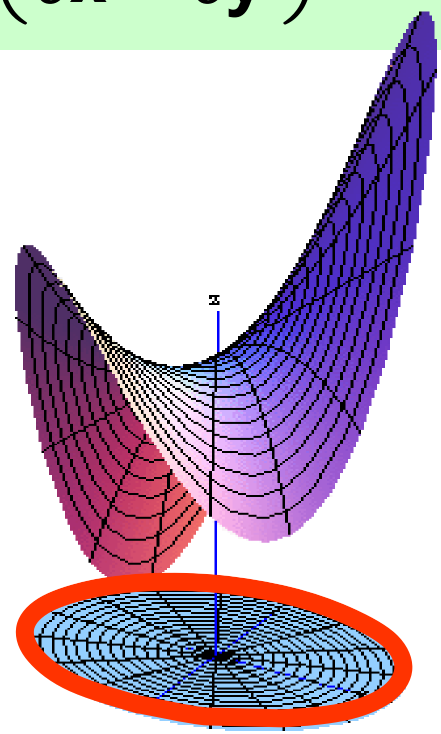
The Gauss-Green Formula

Let R be a region in the xy-plane whose boundary is parameterized by (x(t),y(t)) for $t_{low} \le t \le t_{high}$. Then the following formula holds :

$$\int_{t_{low}}^{t_{high}} \Big(m\big(x(t), y(t)\big) x'(t) + n\big(x(t), y(t)\big) y'(t) \Big) dt = \iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \right) dA$$

With the proper interpretation, we can use this formula to help us compute flow along/across measurements!!

The basic interpretation of Gauss-Green is that it is a correspondence between a line integral of a closed curve and a double integral of the interior region of the closed curve. So for this to work correctly, we just need to make sure the vector field has no singularities in the interior region!



Measuring the Flow of a Vector Field ALONG a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ALONG*** the closed curve is measured by:

$$\int_a^b \mathbf{Field(x(t),y(t)) \bullet (x'(t),y'(t))dt}$$

$$= \int_a^b \Big(\mathbf{m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t)} \Big) \mathbf{dt}$$

$$= \oint_C \mathbf{m(x,y)dx + n(x,y)dy}$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}} - \frac{\partial \mathbf{m}}{\partial \mathbf{y}} \right) \mathbf{dx\,dy}$$

$$\mathbf{Let\ rotField(x,y)} = \frac{\partial \mathbf{n}}{\partial \mathbf{x}} - \frac{\partial \mathbf{m}}{\partial \mathbf{y}}.$$

$$= \iint_R \mathbf{rotField(x,y)\ dx\,dy}$$

Summary: The Flow of A Vector Field ALONG a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_C m(x,y)\,dx + n(x,y)\,dy = \iint_R \text{rotField}(x,y)\;dx\;dy$$

This measures the net flow of the vector field ***ALONG*** the closed curve.

We define the rotation of the vector field as:

$$\text{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = D[n[x,y],x] - D[m[x,y],y]$$

Remembering the Formula for the Rotation of the Vector Field

You can think of the rotation of your vector field as the determinant of a 2x2 matrix with the first row as the differential operator and the second row as your vector field, Field(x,y) = (m(x,y),n(x,y)):

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ m & n \end{vmatrix} = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = \mathbf{rotField(x,y)}$$

Measuring the Flow of a Vector Field ACROSS a Closed Curve

Let C be a closed curve with a counterclockwise parameterization. Then the net flow of the vector field ***ACROSS*** the closed curve is measured by:

$$\int_a^b \mathbf{Field}(x(t), y(t)) \bullet (y'(t), -x'(t))\,dt$$

$$= \int_a^b \left(-n(x(t), y(t))x'(t) + m(x(t), y(t))y'(t)\right)dt$$

$$= \oint_C -n(x, y)dx + m(x, y)dy$$

Let region R be the interior of C. If the vector field has no singularities in R, then we can use Gauss-Green:

$$= \iint_R \left(\frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}\right) dx\,dy$$

$$\text{Let } \mathbf{divField}(x, y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}.$$

$$= \iint_R \mathbf{divField}(x, y)\,dx\,dy$$

Summary: The Flow of A Vector Field ACROSS a Closed Curve:

Let C be a closed curve parameterized counterclockwise. Let Field(x,y) be a vector field with no singularities on the interior region R of C. Then:

$$\oint_C -n(x,y)dx + m(x,y)dy = \iint_R \text{divField}(x,y)\, dx\, dy$$

This measures the net flow of the vector field ***ACROSS*** the closed curve.

We define the divergence of the vector field as:

$$\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = D[m[x,y],x] + D[n[x,y],y]$$

We Are Saving Singularities for Tomorrow

Using Gauss-Green in this way is an amazing computational tool provided our closed curves do not encapsulate any singularities. So for today, we'll explore this material with no singularities.

After this, we'll see how singularities spice things up a bit…

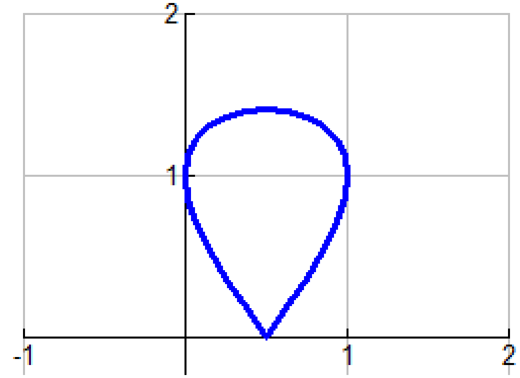
Example 1: Avoiding Computation Altogether

Let $\text{Field}(x,y) = (7x+2, y-6)$ and let C be a closed curve given by

$C(t) = (x(t), y(t)) = (\sin^2(t), \cos(t)+\sin(t))$ for $-\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

Is the net flow of the vector field across the curve from inside to outside or outside to inside?

$$\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 7 + 1 = 8$$



$$\oint_C -n(x,y)dx + m(x,y)dy = \iint_R \text{divField}(x,y)\, dx\, dy = \iint_R 8\, dx\, dy > 0$$

Since divField(x,y) is ALWAYS positive for all (x,y) and there are no singularities for any (x,y), this integral is positive for any closed curve.

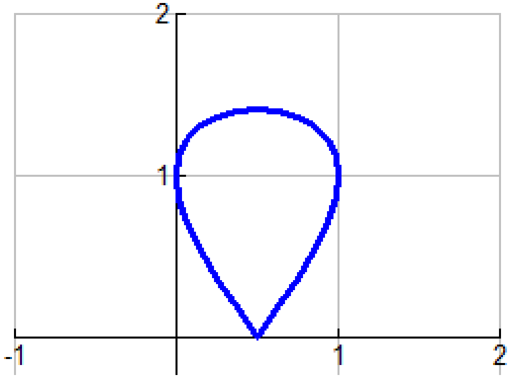
That is, for ANY closed curve, the net flow of the vector field across the curve is from inside to outside.

Example 1: Avoiding Computation Altogether

Let $\text{Field}(x,y) = (7x+2, y-6)$ and let C be a closed curve given by

$C(t) = (x(t), y(t)) = (\sin^2(t), \cos(t)+\sin(t))$ for $-\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

Is the net flow of the vector field across the curve from inside to outside or outside to inside?



When divField(x,y)>0, this is a source of new fluid. Adding up all sources will give you net flow of the vector field across the curve from inside to outside.

This method was nice because it was almost no computation!

Summary: The Divergence Locates Sources and Sinks

Let C be a closed curve with a counterclockwise parameterization with ***no singularities*** on the interior of the curve. Then:

If divField(x,y)>0 for all points in C, then all these points are sources and the net flow of the vector field across C is from inside to outside.

If divField(x,y)<0 for all points in C, then all of these points are sinks and the net flow of the vector field across C is from outside to inside.

If divField(x, y) = 0 for all points in C, then the net flow of the vector field across C is 0. When this happens for all the points in your vector field, it is called incompressible.

Summary: The Rotation Helps You Find Clockwise/Counterclockwise Swirl

Let C be a closed curve with a counterclockwise parameterization with ***no singularities*** on the interior of the curve. Then:

If rotField(x,y)>0 for all points in C, then all these points add counterclockwise swirl and the net flow of the vector field along C is counterclockwise.

If rotField(x,y)<0 for all points in C, then all of these points add clockwise swirl and the net flow of the vector field along C is clockwise.

If rotField(x, y) = 0 for all points in C, then these points have no swirl, and the net flow of the vector field along C is 0. When this happens for all points in the vector field, your vector field is called conservative or irrotational.

You Try One!

Let $Field(x,y) = \left(y^5, -2x^3\right)$ and let C be a closed curve given by

$C(t) = (x(t), y(t)) = \left(\sin^2(t) + \cos(t), \sin(t) + \cos(t)\right)$ for $\frac{\pi}{2} \le t \le 2\pi.$

Is the net flow of the vector field along the curve counterclockwise or clockwise?

$$rotField(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = -6x^2 - 5y^4 < 0$$

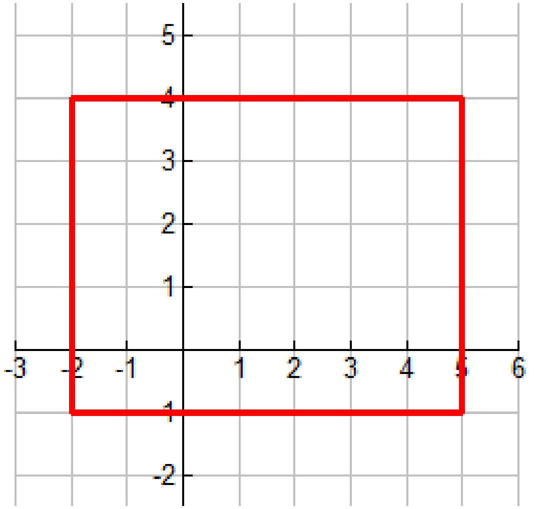
$$\oint_C m(x,y)dx + n(x,y)dy = \iint_R rotField(x,y)\ dx\ dy < 0$$

Since rotField(x,y) is ALWAYS negative for all (x,y) and there are no singularities for any (x,y), this integral is negative for any closed curve.

That is, for ANY closed curve, the net flow of the vector field along the curve is clockwise.

Example 2: Find the Net Flow of a Vector Field *ACROSS* Closed Curve

Let $\text{Field}(x,y) = \left(x^2 - 2xy, -y^2 + x\right)$ and let C be the rectangle bounded by $x = -2,\ x = 5,\ y = -1,$ and $y = 4$. Measure the net flow of the vector field across the curve.



$$\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 2x - 4y$$

$$\oint_C -n(x,y)dx + m(x,y)dy = \iint_R \text{divField}(x,y)\ dx\ dy$$

$$= \int_{-1}^{4}\int_{-2}^{5} \left(2x - 4y\right) dx\ dy$$

$$= -105$$

Negative. The net flow of the vector field across our closed curve is from outside to inside.

Why Does Gauss-Green Make Intuitive Sense When Measuring Flow?

$$\oint_C -n(x,y)\,dx + m(x,y)\,dy = \iint_R \text{divField}(x,y)\,dx\,dy$$

To compute the net flow of water across the orange curve, we could either measure how much water is going past the orange curve at each point (single integral)… OR we could accumulate the net effect of all of the sources and sinks inside the curve (the double integral). Either way, we get the same measurement!

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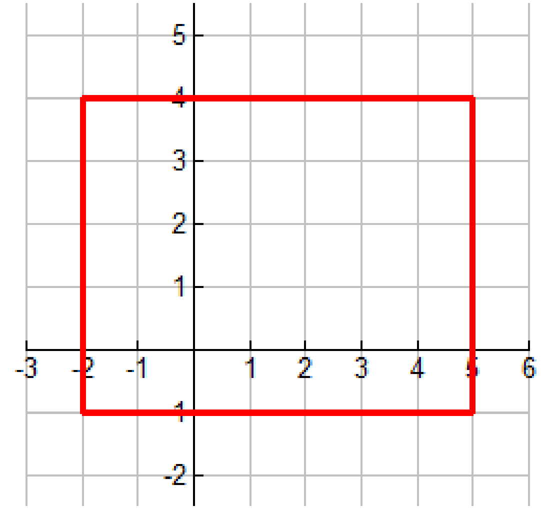
A Variation on This Analogy

You can measure how much water is spilling out of an overflowing tub either by measuring around the rim of the tub (single path integral) or looking at the net effect of the faucet/drain (double integral with sources/sinks)…

You Try One: Find the Net Flow of a Vector Field *ALONG* a Closed Curve

Let $\text{Field}(x,y)=\left(x^2-2xy,-y^2+x\right)$ and let C be the rectangle bounded by $x=-2,\ x=5,\ y=-1,$ and $y=4$. Measure the net flow of the vector field along the curve.



$$\text{rotField}(x,y)=\frac{\partial n}{\partial x}-\frac{\partial m}{\partial y}=1+2x$$

$$\oint_C m(x,y)dx+n(x,y)dy=\iint_R \text{rotField}(x,y)\ dx\ dy$$

$$=\int_{-1}^{4}\int_{-2}^{5}\left(1+2x\right)dx\ dy$$

$$=140$$

Positive. The net flow of the vector field along our closed curve is counterclockwise.

Updating the Gradient Test:

Last Chapter :

$$1)\ \frac{\partial n}{\partial x} = \frac{\partial m}{\partial y}$$

2) No singularities

How can we use our new terminology in the gradient test?

This Chapter :

$$1)\ \text{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 0$$

2) No singularities

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Summary: The Flow of a Gradient Field Along a Closed Curve

Let $\text{Field}(x,y) = \big(m(x,y), n(x,y)\big)$ be a gradient field, and let C be a simple closed curve with a parameterization (x(t),y(t)) for $a \le t \le b$.

1) $\int_a^b \text{Field}(x(t), y(t)) \bullet (x'(t), y'(t))dt = 0$

2) $\oint_C m(x,y)dx + n(x,y)dy = 0$

3) The net flow of a gradient field along a simple closed curve is 0.

Note: This is only true where the vector field is defined! If the vector field is undefined inside your closed curve, this doesn't work! Today we'll deal with singularities since they mess up all of your gradient intuition.

Why is this intuitively true?

How do we know a closed curve can't be a trajectory of a gradient field?

Is the flow of a gradient field ACROSS a closed curve 0?

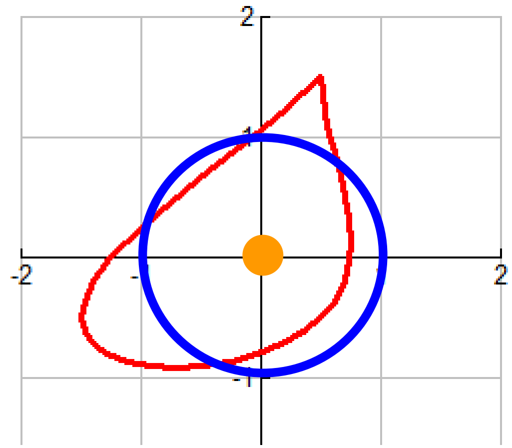
Much Ado About Singularities

All of our calculations yesterday were for curves that did not contain any singularities. Today, we will talk about how to cope with vectors fields that have singularities!

Example 1: A Flow Along Measurement With a Singularity

Let $\text{Field}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ and let C be the closed curve described by

$C(t) = \left(\sin^2(t) + \cos(t) - \frac{1}{2}, \cos(t) + \sin(t) + \frac{1}{2} \right)$ for $\frac{\pi}{2} \le t \le 2\pi$. Compute the net flow of the vector field along the curve.



$$\text{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 0$$

Since rotField(x,y)=0, your field behaves like a gradient away from singularities.

$\Rightarrow$ The only swirl can come from singularities!

There is a singularity at (0,0).

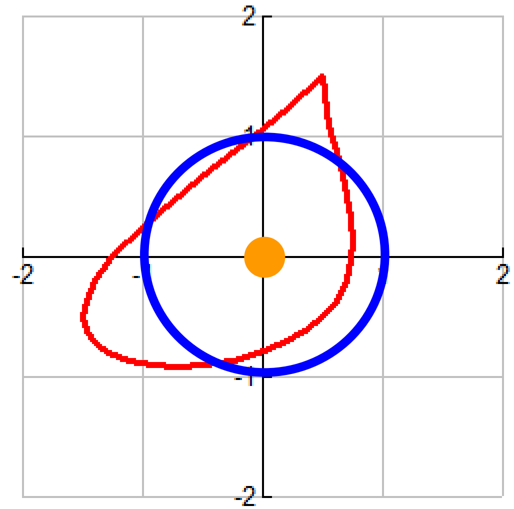
We can replace our curve with any curve that encapsulates the singularity:

$$C_2(t) = (\cos(t), \sin(t)) \text{ for } 0 \le t \le 2\pi$$

Note: Had there been no singularities in the curve, how would we know that the net flow of the vector field ALONG the curve would be 0?

Example 1: A Flow Along Measurement With a Singularity

$$\text{Field}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \text{ and } C(t) = \left(\cos(t), \sin(t) \right) \text{ for } 0 \le t \le 2\pi$$



Because of the singularity, we can't use $\iint \text{rotField}(x,y)\,dx\,dy$**. Instead, we will need to compute** $\oint_C m(x,y)dx + n(x,y)dy$ **the old-fashioned way:**

$$\int_0^{2\pi} \text{Field}(x(t),y(t)) \bullet (x'(t),y'(t))\,dt$$

$$= \int_0^{2\pi} \left(\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2(t)} \right) \bullet (-\sin(t),\cos(t))\,dt$$

$$= \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

So the net flow of the vector field along the curve is counterclockwise!

Example 2: A Flow Along With Multiple Singularities? No Problem!

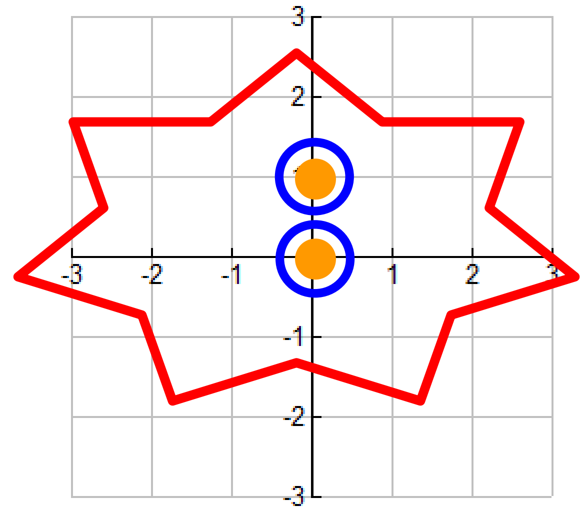
Let $\text{Field}(x,y) = \left(\frac{-y}{x^2+y^2} + \frac{y-1}{x^2+(y-1)^2}, \frac{x}{x^2+y^2} - \frac{x}{x^2+(y-1)^2} \right)$ and let C be the curve pictured below. Compute the flow of the vector field along the curve.

$$\text{rotField}(x,y) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 0$$

Since rotField(x,y)=0, your field behaves like a gradient away from singularities.

$\Rightarrow$ The only swirl can come from singularities!

There are singularities at (0,0) and (0,1).



We can encapsulate the singularities with two little circles and sum our results!

$C_1(t) = 0.5\left(\cos(t), \sin(t)\right)$ for $0 \le t \le 2\pi$

$C_2(t) = 0.5\left(\cos(t), \sin(t)\right) + (0,1)$ for $0 \le t \le 2\pi$

Example 2: A Flow Along With Multiple Singularities? No Problem!

Let $\text{Field}(x,y) = \left(\frac{-y}{x^2+y^2} + \frac{y-1}{x^2+(y-1)^2}, \frac{x}{x^2+y^2} - \frac{x}{x^2+(y-1)^2} \right)$ and let C be the curve pictured below. Compute the flow of the vector field along the curve.

```
Clear[m, n, x, y, a];
m[x_, y_] = -y/(x^2 + y^2) - (y - 1)/(x^2 + (y - 1)^2);
n[x_, y_] = x/(x^2 + y^2) + x/(x^2 + (y - 1)^2);
{x1[t_], y1[t_]} = {0, 0} + .5 * {Cos[t], Sin[t]};
{x2[t_], y2[t_]} = {0, 1} + .5 * {Cos[t], Sin[t]};
NIntegrate[m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, 0, 2 π}] + NIntegrate[m[x2[t], y2[t]] x2'[t] + n[x2[t], y2[t]] y2'[t], {t, 0, 2 π}]
12.5664
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Summary: Flow Along When rotField(x,y)=0

Let $\text{rotField}(x,y) = \dfrac{\partial n}{\partial x} - \dfrac{\partial m}{\partial y} = 0$. Here are some conclusions about the net flow of the vector field along various closed curves:

If C doesn't contain any singularities, then $\oint_C m(x,y)dx + n(x,y)dy = 0.$

If C contains a singularity, then $\oint_C m(x,y)dx + n(x,y)dy = \oint_{C_1} m(x,y)dx + n(x,y)dy$ for any substitute curve C_1 containing the same singularity (and no new extras).

If C contains k singularities, then

$$\oint_C m(x,y)dx + n(x,y)dy = \oint_{C_1} m(x,y)dx + n(x,y)dy + \ldots + \oint_{C_k} m(x,y)dx + n(x,y)dy$$

for little circles, $C_1, \ldots, C_k$, encapsulating each of these singularities.

Summary: Flow Across When divField(x,y)=0

Let $\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 0$. Here are some conclusions about the net flow of the vector field across various closed curves:

If C doesn't contain any singularities, then $\oint_C -n(x,y)dx + m(x,y)dy = 0$.

If C contains a singularity, then $\oint_C -n(x,y)dx + m(x,y)dy = \oint_{C_1} -n(x,y)dx + m(x,y)dy$ for any substitute curve C_1 containing the same singularity (and no new extras).

If C contains k singularities, then

$$\oint_C -n(x,y)dx + m(x,y)dy = \oint_{C_1} -n(x,y)dx + m(x,y)dy + \ldots + \oint_{C_k} -n(x,y)dx + m(x,y)dy$$

for little circles, $C_1, \ldots, C_k$, encapsulating each of these singularities.

Differential Operators from Calc A:

We can think of $\frac{d}{dx}$ as a differential operator that tell us "take the derivative with respect to x":

$$y = e^{2x}$$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(e^{2x}\right)$$

$$\frac{dy}{dx} = 2e^{2x}$$

Differential Operators from Calc A:

This might help you understand why calculus teachers say that the second derivative is $\frac{d^2y}{dx^2}$:

$$y'' = \frac{d}{dx}\left(\frac{d}{dx}(y)\right)$$

$$= \left(\frac{d}{dx}\right)^2 (y)$$

$$= \frac{d^2}{dx^2}(y)$$

$$= \frac{d^2y}{dx^2}$$

The Gradient: A New Perspective

Let ∇ be the differential operator, named "del."

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

Let f(x, y) be a function. Then the gradient vector is defined as follows:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

The Gradient: A New Perspective

Consider $\nabla \bullet \nabla$:

$$\nabla \bullet \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \bullet \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Now consider $\nabla \bullet \nabla f$, which we will call the <u>Laplacian</u> :

$$\nabla \bullet \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

New symbol for the Laplacian: $\nabla \bullet \nabla f = \Delta f$.

$$\Delta f = \nabla \bullet \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Connecting the Laplacian and the Divergence of the Gradient Field:

Let $z = f(x,y)$ and let $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(m(x,y), n(x,y)\right)$

$$\text{divField}(x,y) = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}$$

$$= \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= \nabla \bullet \nabla f$$

$$= \Delta f$$

So the Laplacian, $\nabla \bullet \nabla f$, is the divergence of the gradient field!

What if $\nabla \bullet \nabla f = 0$?

What does it mean if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0?$$

Connecting the Laplacian and the Divergence of the Gradient Field:

The Laplacian of f(x,y) is the divergence of the gradient of f(x,y). There are three main notations for this: $\Delta f = \nabla \bullet \nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

If $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0,$ then the divergence of the gradient field of f(x,y) is 0. This means that the gradient field has no sources or sinks, which means the surface f(x,y) has no maxes or mins.

Connecting the Laplacian and the Divergence of the Gradient Field:

Which surfaces could be $z = f(x, y)$ if we are given $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$?