Lesson 9

Change of Variables to Compute Double Integrals

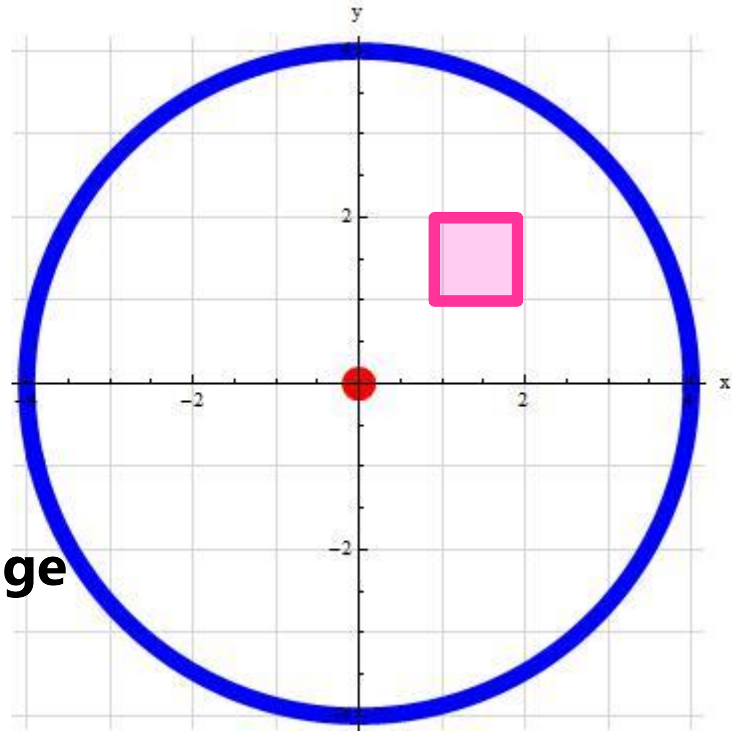
Example 1: A Familiar Double Integral

Use a double integral to calculate the area of a circle of radius 4 centered at the origin:

$$\iint_R 1\,dA$$

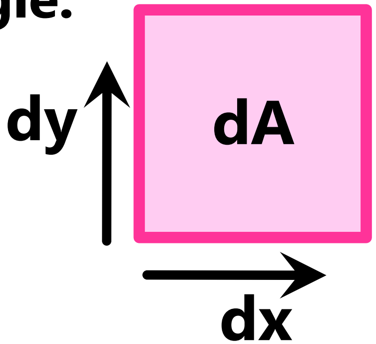
The region R is given by $x^2 + y^2 = 16$:

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1\,dA$$



Now dA means "a small change in area" in xy-coordinates. We can measure a small change in area with a little rectangle.

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1\,dy\,dx$$



So dA = dy dx:

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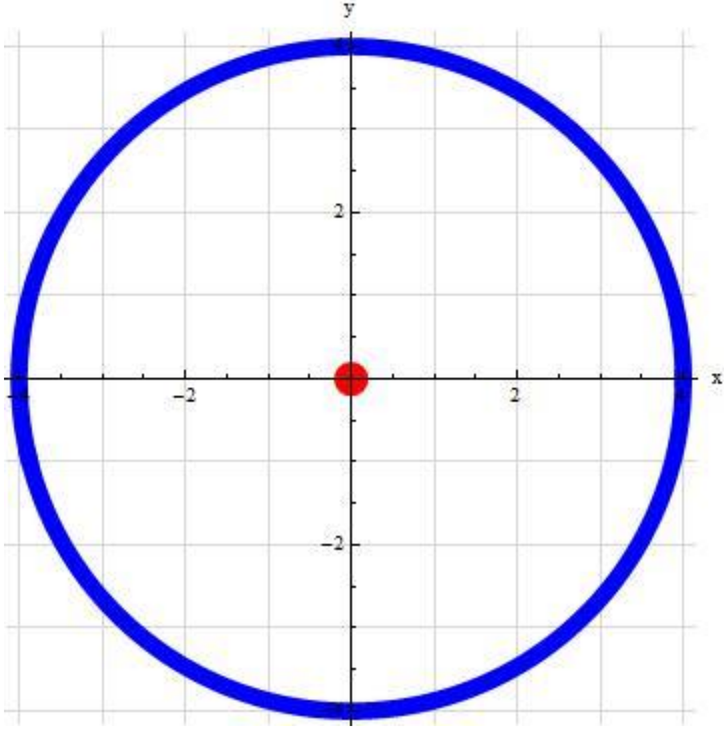
Example 1: A Familiar Double Integral

This integral is actually pretty complicated to evaluate:

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1\,dy\,dx = \int_{-4}^{4}\left(2\sqrt{16-x^2}\right)dx$$

$$= \left[\sqrt{16-x^2} + 16\sin^{-1}\left(\frac{x}{4}\right)\right]_{-4}^{4}$$

$$= 16\pi$$



What a mess! There must be a better way to use integration to do this…

What coordinate system would you prefer to use to when dealing with a region like R?

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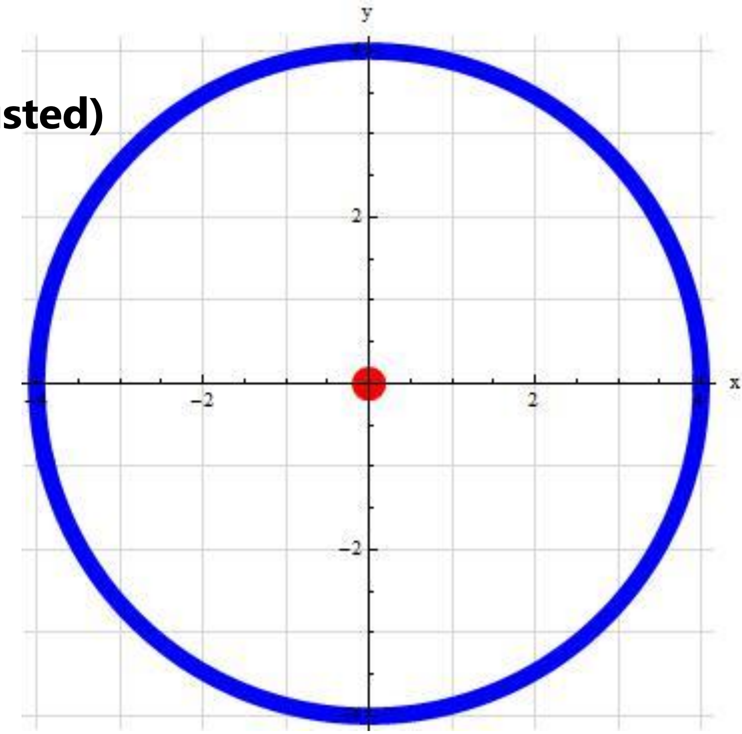
Example 2: An Attempt at a Change of Variables

Right! Let's try polar coordinates:

$$\begin{cases} x(r,t) = r\cos(t) \\ y(r,t) = r\sin(t) \end{cases} \qquad \begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ t(x,y) = \tan^{-1}\left(y/x\right) \text{ (quadrant adjusted)} \end{cases}$$



In this coordinate system, the region R is described quite elegantly with $0 \le r \le 4$ and $0 \le t \le 2\pi$. We will try to (naively) apply this to our double integral:

$$\iint_R 1\,dA = \int_0^{2\pi}\int_0^4 1\,{\color{red}dr\,dt}$$

$$= \int_0^{2\pi} \left[r\right]_0^4 {\color{red}dt}$$

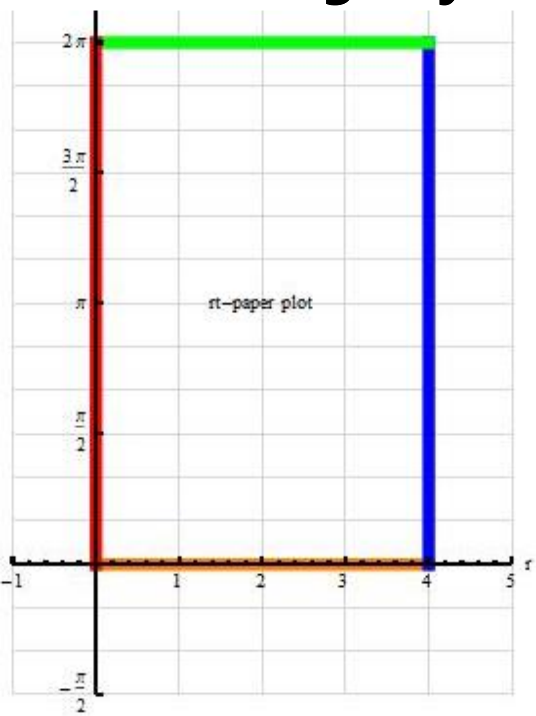
$$= 8\pi$$

Oh no! 8π is not the right answer!! We wanted 16π...

Detour: Working Out the Change of Variables the Right Way

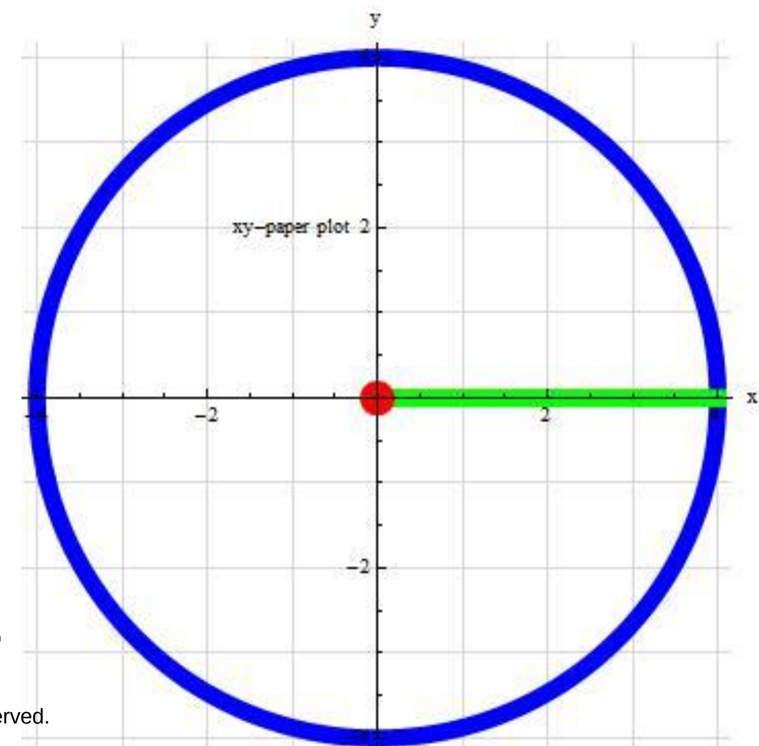
Our misconception was that dA = dr dt. Our dA refers to a small change in xy-area, not rt-area. We need to delve a bit deeper to sort this out:

For $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1\,dy\,dx$ **our region was a circle, while** $\int_{0}^{2\pi}\int_{0}^{4} 1\,dr\,dt$ **looks more like the integral you'd take for a rectangle:**



$$x(r,t)=r\cos(t)$$
$$y(r,t)=r\sin(t)$$
→

$$r(x,y)=\sqrt{x^2+y^2}$$
$$t(x,y)=\tan^{-1}(y/x)$$
←



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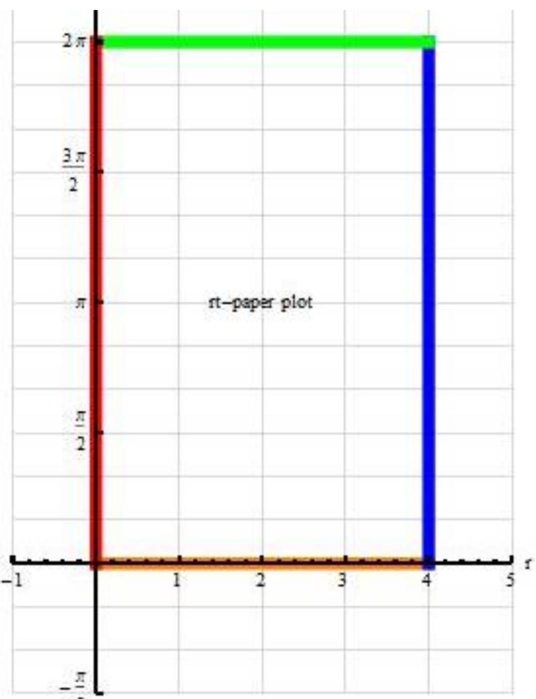
Detour: Working Out the Change of Variables the Right Way

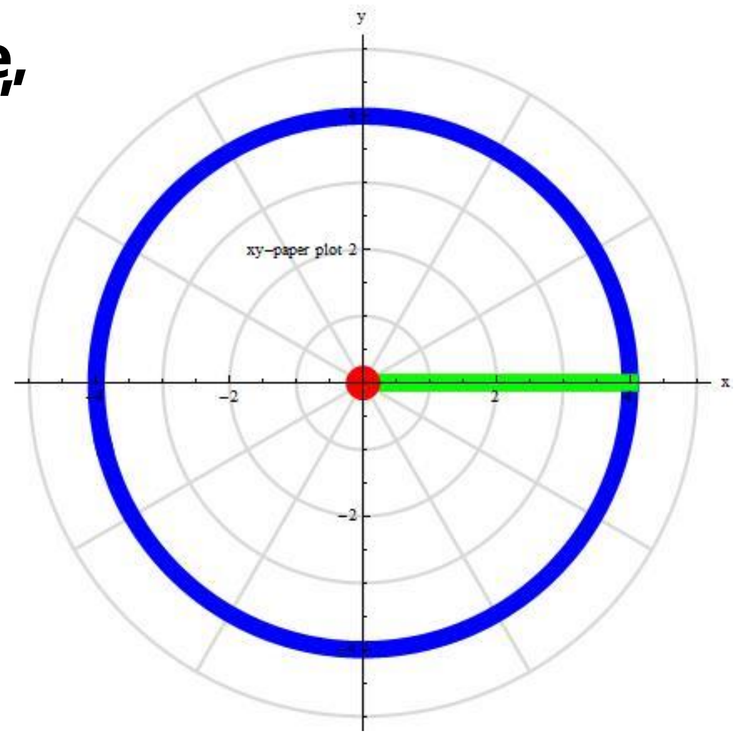
Let's analyze the transformation $T(r,t) = \left(r\cos t, r\sin t\right)$ piece by piece and verify that the picture shown below makes sense:



Let's map a few points just for practice:

$x(r,t) = r\cos t$

$y(r,t) = r\sin t$

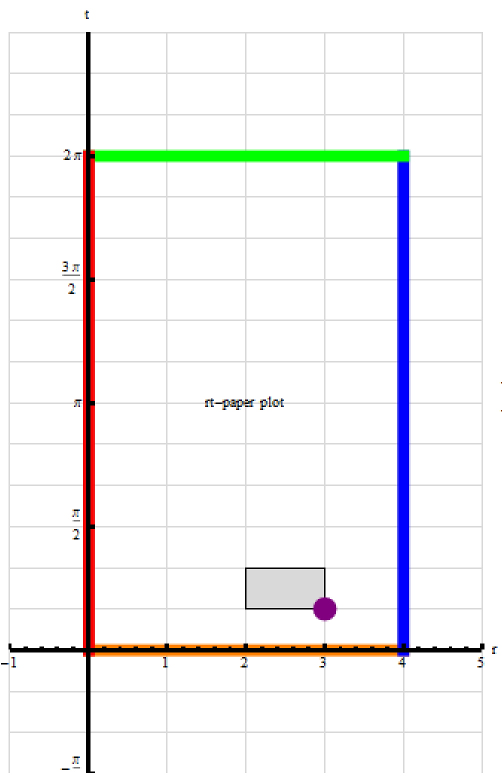


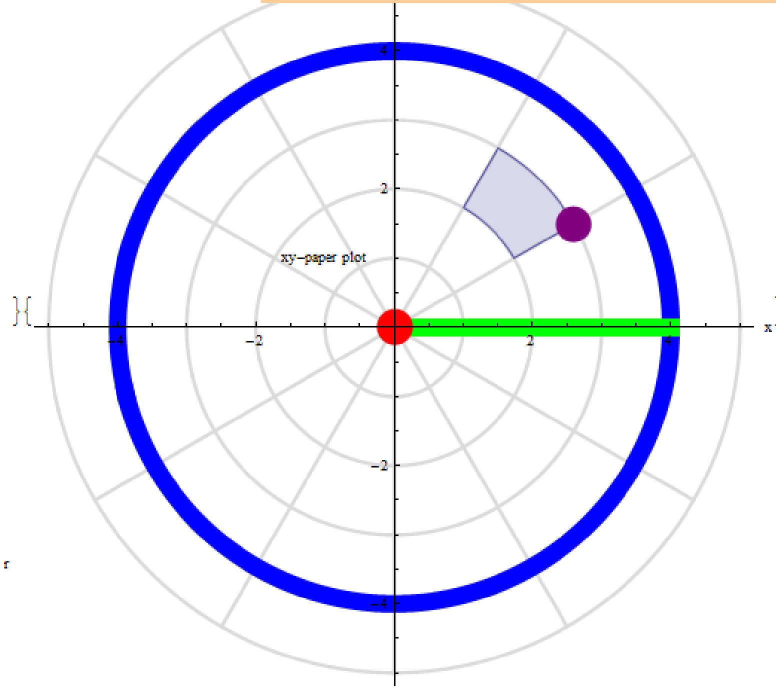
This whole process can be called a mapping, a transformation, a change of variables, or a change of coordinates.

Detour: Working Out the Change of Variables the Right Way

Try the Mathematica applet I made to see how a small region in rt-space maps into xy-space:

This is called a non-area-preserving map.



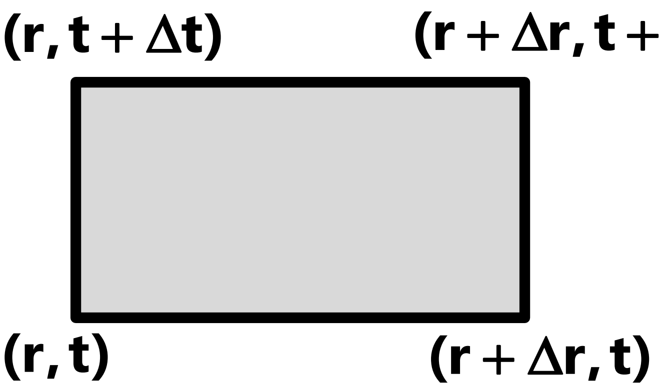


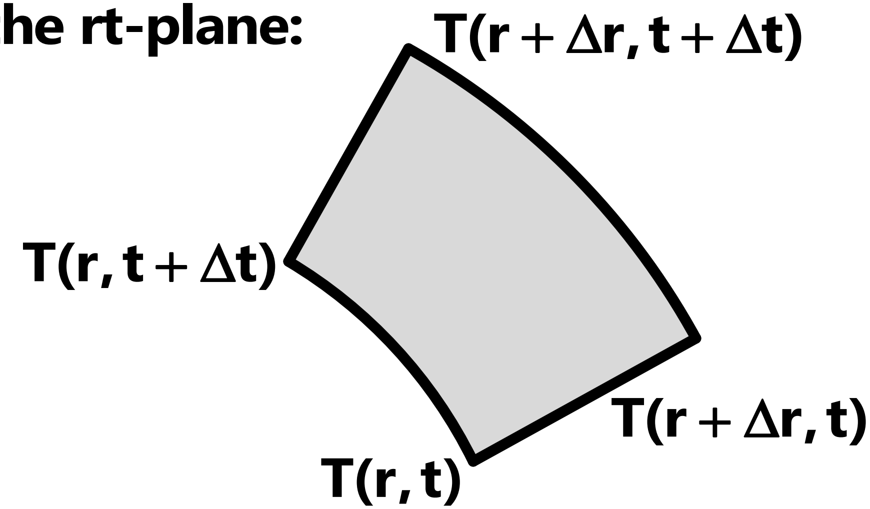
rt-rectangles map into xy-sectors. Notice that this size of the sectors varies while the size of the rectangles doesn't…

Now we are ready to start thinking about how a small change in area on the rectangular region relates to a change in area on the circular region.

Detour: Working Out the Change of Variables the Right Way

Now imagine a very small rectangle in the rt-plane:





That is, consider the limit as ΔA, Δr, and Δt tend to 0:

$$T(r+\Delta r, t) - T(r,t) = \frac{T(r+\Delta r, t) - T(r,t)}{\Delta r}\Delta r$$

$$\approx \frac{\partial T}{\partial r}\Delta r$$

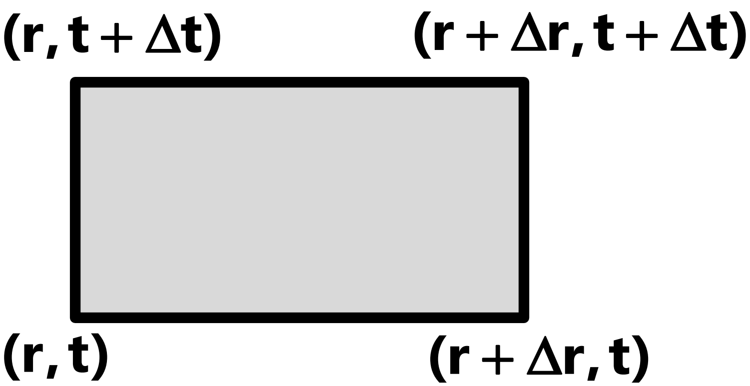
$$T(r, t+\Delta t) - T(r,t) = \frac{T(r, t+\Delta t) - T(r,t)}{\Delta t}\Delta t$$

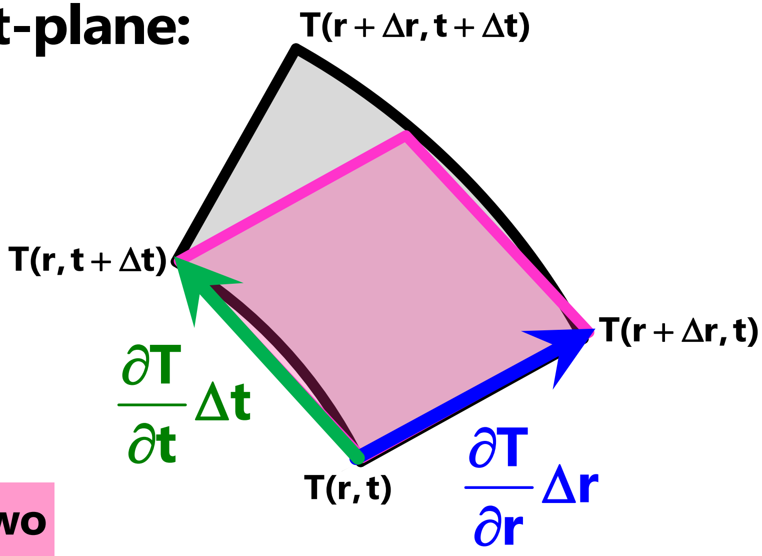
$$\approx \frac{\partial T}{\partial t}\Delta t$$

These approximations using partial derivatives let us approximate our sector using a parallelogram:

Detour: Working Out the Change of Variables the Right Way

Now imagine a very small rectangle in the rt-plane:





Theorem: The area of the parallelogram formed by two vectors V and W equals $|V \times W|$.

$$\Delta A \approx \left|\frac{\partial T}{\partial r}\Delta r \times \frac{\partial T}{\partial t}\Delta t\right| = \left|\frac{\partial T}{\partial r} \times \frac{\partial T}{\partial t}\right| \Delta r \Delta t$$

(We can take a scalar out of either vector in a cross product)

$$= \begin{vmatrix} i & j & k \\ \frac{\partial T_1}{\partial r} & \frac{\partial T_2}{\partial r} & 0 \\ \frac{\partial T_1}{\partial t} & \frac{\partial T_2}{\partial t} & 0 \end{vmatrix} \Delta r \Delta t = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_2}{\partial r} \\ \frac{\partial T_1}{\partial t} & \frac{\partial T_2}{\partial t} \end{vmatrix} \Delta r \Delta t$$

The Area Conversion Factor:

If we let Δr and Δt tend to zero, we can use this to get

$$dA = \left| \begin{matrix} \dfrac{\partial T_1}{\partial r} & \dfrac{\partial T_2}{\partial r} \\ \dfrac{\partial T_1}{\partial t} & \dfrac{\partial T_2}{\partial t} \end{matrix} \right| dr\,dt$$

$\begin{bmatrix} \dfrac{\partial T_1}{\partial r} & \dfrac{\partial T_2}{\partial r} \\ \dfrac{\partial T_1}{\partial t} & \dfrac{\partial T_2}{\partial t} \end{bmatrix}$ is called the Jacobian matrix, and $\left| \begin{matrix} \dfrac{\partial T_1}{\partial r} & \dfrac{\partial T_2}{\partial r} \\ \dfrac{\partial T_1}{\partial t} & \dfrac{\partial T_2}{\partial t} \end{matrix} \right|$ is called the Jacobian determinant.

You can let $A_{xy}(r,t) = \left| \begin{matrix} \dfrac{\partial T_1}{\partial r} & \dfrac{\partial T_2}{\partial r} \\ \dfrac{\partial T_1}{\partial t} & \dfrac{\partial T_2}{\partial t} \end{matrix} \right|$. Think of the Jacobian determinant as an Area Conversion Factor that lets us compute an xy-space integral in rt-space:

$$\text{Hence, } \iint_{R_{xy}} f(x,y)\,dA = \iint_{R_{rt}} f(x(r,t), y(r,t))\,\left|A_{xy}(r,t)\right|\,dr\,dt$$

The Area Conversion Factor:

$$\iint_{R_{xy}} f(x,y)\,dA = \iint_{R_{rt}} f(x(r,t),y(r,t))\,\left|A_{xy}(r,t)\right|\,dr\,dt$$

Let $T(r,t)$ be a transformation from rt-space to xy-space.

That is, $T(r,t) = \left(T_1(r,t), T_2(r,t)\right) = (x(r,t), y(r,t))$.

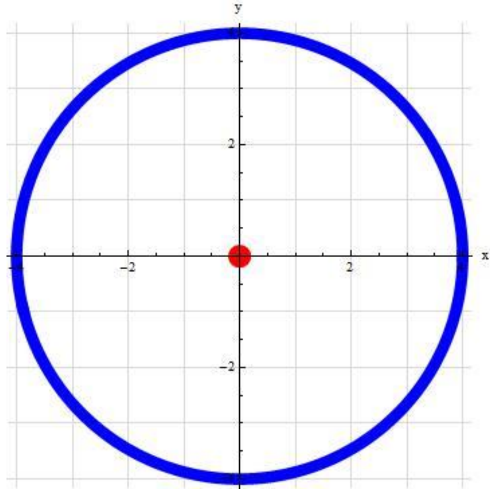
Then $A_{xy}(r,t) = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_2}{\partial r} \\ \frac{\partial T_1}{\partial t} & \frac{\partial T_2}{\partial t} \end{vmatrix}$.

Note: Since we derived $A_{xy}(r,t)$ as the magnitude of a cross product, we need it to be positive. This is why we put absolute value bars into the formula above.

Example 3: Fixing Example 2

Remember we wanted $\iint_R 1\, dA$ for a circle of radius 4 centered at (0,0).

$$T(r,t) = (x(r,t), y(r,t)) = (r\cos(t), r\sin(t))$$



$$A_{xy}(r,t) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(t) & \sin(t) \\ -r\sin(t) & r\cos(t) \end{vmatrix}$$

$$= r\cos^2(t) + r\sin^2(t)$$

$$= r\left(\cos^2(t) + \sin^2(t)\right)$$

$$= r$$

$$\iint_R 1\, dA = \int_0^{2\pi}\int_0^4 1\, r\, dr\, dt$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^4 dt$$

$$= \int_0^{2\pi} 8\, dt$$

$$= 16\pi$$

Phew! We did it!

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Summary of Change of Variables for Polar Coordinates

$$T(r,t) = (x(r,t), y(r,t)) = (r\cos(t), r\sin(t))$$

$$A_{xy}(r,t) = r$$

$$\iint_{R_{xy}} f(x,y)\,dA = \iint_{R_{rt}} f(x(r,t), y(r,t))\; r\; dr\; dt$$

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Example 4: More With Polar Coordinates

- By now you probably already asked yourself why this change of variables is useful. This was just a circle of course! We could have used $A = \pi r^2$ or the Gauss-Green formula.

- Hence, we should look at an example where a double integral in xy-coordinate space would be horribly messy, the boundary region is hard to parameterize for Gauss-Green, and we can't just plug into a familiar area formula…

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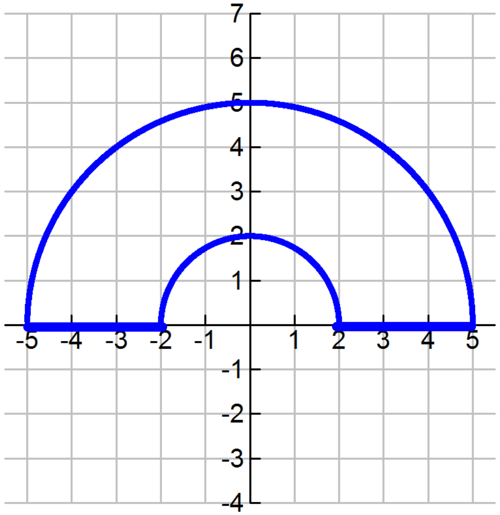
Example 4: More With Polar Coordinates

Compute $\iint_R x^2 + y^2 \, dA$ for the region satisfying the following inequalities:

$$x^2 + y^2 \le 25,\ x^2 + y^2 \ge 4, y \ge 0$$

This would have been HORRIBLE in rectangular coordinates. But using polar coordinates, $x = r\cos(t)$ and $y = r\sin(t)$ this is the rt-rectangle with $0 \le t \le \pi$ and $2 \le r \le 5$:



$$\iint_R x^2 + y^2 \, dA = \int_0^{\pi}\int_2^{5}\left(\left(r\cos(t)\right)^2 + \left(r\sin(t)\right)^2\right)\bullet r\,dr\,dt$$

$$= \int_0^{\pi}\int_2^{5} r^3 \, dr\,dt$$

$$= \int_0^{\pi} \frac{609}{4}\, dt$$

$$= \frac{609}{4}\pi$$

Substitute $x = r\cos(t)$ and $y = r\sin(t)$

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Analogy Time:

This whole change of variables thing isn't just for polar coordinates.

We can change from xy-space to any uv-coordinate space we want:

$$\iint_{R_{xy}} f(x,y)\,dA = \iint_{R_{uv}} f(x(u,v),y(u,v))\,\left|A_{xy}(u,v)\right|\,du\,dv$$

This should be reminiscent of change of variables for single-variable calculus:

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\,x'(u)\,du$$

So you can think of the Jacobian Determinant, $\left|A_{xy}(u,v)\right|$, as a higher-dimensional analogue of $x'(u)$ in our old friend, u-substitution.

Let's try one in a single variable:

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Example 5: Change of Variables for Single Variable Calculus

Compute $\int_{e^2}^{e^6} \frac{\ln(x)}{x}\, dx.$

$$= \int_2^6 f(x(u))\, x'(u)\, du$$

$$= \int_2^6 f(e^u)\, e^u\, du$$

$$= \int_2^6 \frac{u}{e^u}\, e^u\, du$$

$$= \left[\frac{u^2}{2}\right]_2^6$$

$$= 16$$

$$\int_{x(a)}^{x(b)} f(x)\, dx = \int_a^b f(x(u))\, x'(u)\, du$$

$$f(x) = \frac{\ln(x)}{x}$$

$$x(u) = e^u$$

$$x'(u) = e^u$$

It works! This is the same as your more familiar method:

$$\int_2^6 u\, du = 16$$

$$u = \ln(x)$$

$$du = \frac{1}{x} dx$$

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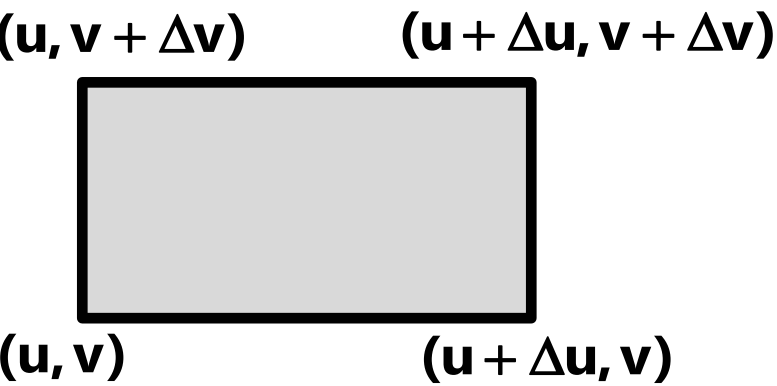
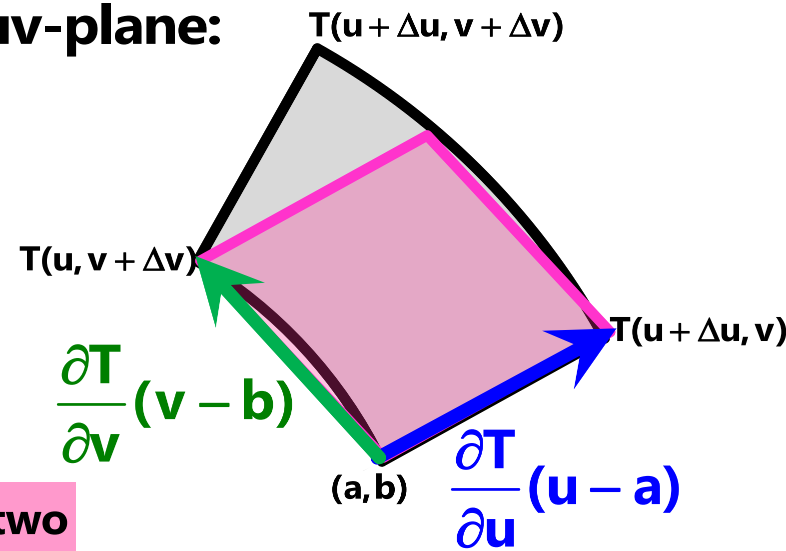
Note for the Literacy Sheet

Now imagine a very small rectangle in the uv-plane:





Theorem: **The area of the parallelogram formed by two vectors V and W equals $|V \times W|$.**

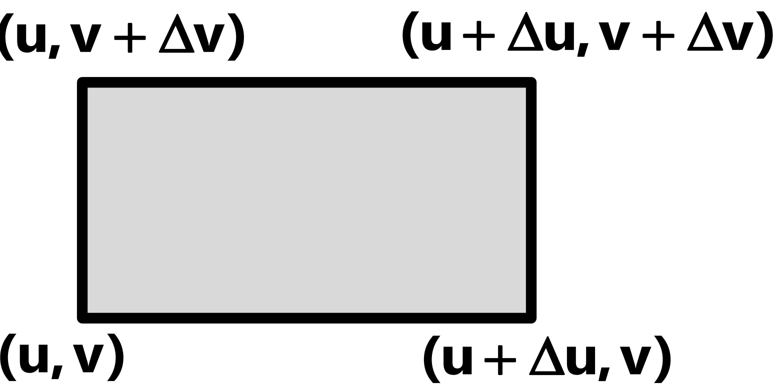
$$\Delta A \approx \left|\frac{\partial T}{\partial u}\Delta u \times \frac{\partial T}{\partial v}\Delta v\right| = \left|\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right| \Delta u \Delta v$$

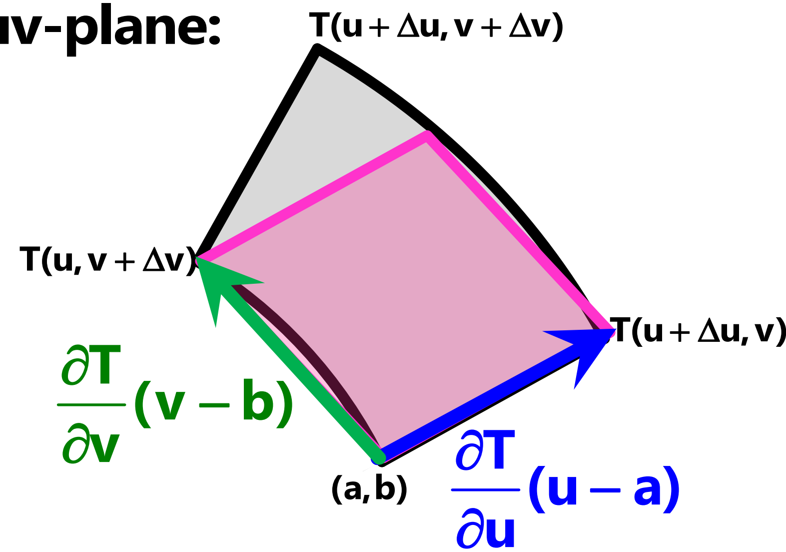
(We can take a scalar out of either vector in a cross product)

$$= \left|\begin{matrix} i & j & k \\ \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & 0 \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & 0 \end{matrix}\right| \Delta u \Delta v = \left|\begin{matrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} \end{matrix}\right| \Delta u \Delta v$$

A Note for the Literacy Sheet

Now imagine a very small rectangle in the uv-plane:



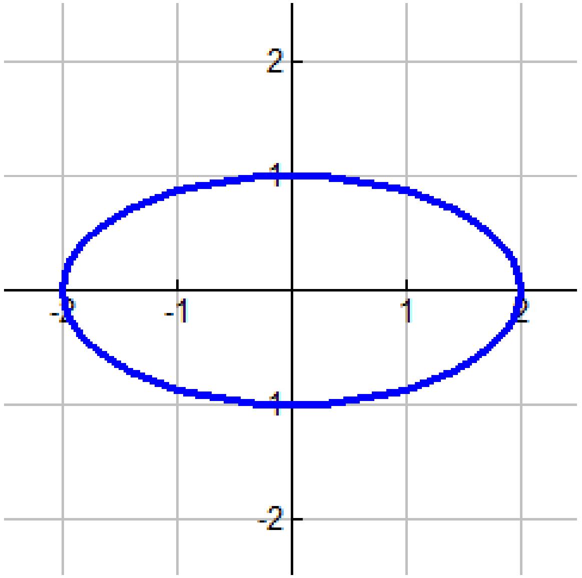


We are using linearizations to approximate the area of the sector by creating a parallelogram. You will be responsible for explaining how this works on your literacy sheet. Make sure you read B.2 and B.3 in the Basics.

Example 6: Beyond Polar Coordinates

Compute $\iint_R y^2 \, dA$ for the region given by the ellipse $\left(\frac{x}{2}\right)^2 + y^2 = 1.$

Let's let $x = 2u\cos(v)$ and $y = u\sin(v)$ for $0 \le v \le 2\pi$ and $0 \le u \le 1$. These are NOT polar coordinates, so we need to compute $A_{xy}(u,v)$:



$$A_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2\cos(v) & \sin(v) \\ -2u\sin(v) & u\cos(v) \end{vmatrix}$$

$$= 2u\cos^2(v) + 2u\sin^2(v)$$

$$= 2u\left(\cos^2(v) + \sin^2(v)\right)$$

$$= 2u$$

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Example 6: Beyond Polar Coordinates

If $x(u,v) = 2u\cos(v)$, then

$$\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right) = \nabla x = \text{gradx}[u,v].$$

Compute $\iint_R y^2 \, dA$ for the region given b

If $y(u,v) = u\sin(v)$, then

$$\left(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}\right) = \nabla y = \text{grady}[u,v].$$

Let's let $x = 2u\cos(v)$ and $y = u\sin(v)$ for $0 \leq$

$0 \leq u \leq 1$. These are NOT polar coordinates, s

to compute $A_{xy}(u,v)$:

$$A_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & \\ -2 & \end{vmatrix}$$

$= 2u\cos^2(v) + 2u\sin^2(v)$

$=$

$=$

So you can think of the area conversion factor as a determinant of a pair of gradient vectors from x(u,v) and y(u,v).

In general, you will start to notice that a change of variables can often be found by parameterizing your curve/surface/etc.

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Example 6: Beyond Polar Coordinates

Compute $\iint_R y^2 \, dA$ for the region given by the ellipse $\left(\frac{x}{2}\right)^2 + y^2 = 1.$

$$\iint_R y^2 \, dA = \int_0^{2\pi}\int_0^1 \left(u^2 \sin^2(v)\right) \bullet 2u \, du \, dv$$

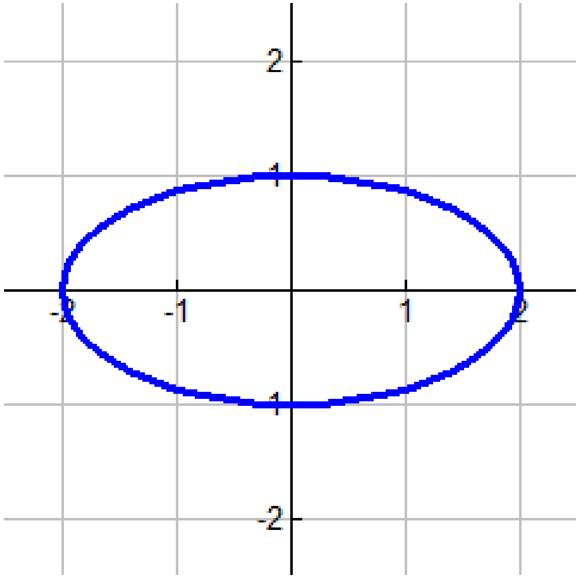
Substitute $x = 2u\cos(v)$ and $y = u\sin(v)$.

$$= 2\int_0^{2\pi}\int_0^1 u^3 \bullet \sin^2(v) \, du \, dv$$

$$= 2\int_0^{2\pi}\left[\frac{u^4}{4} \bullet \sin^2(v)\right]_{u=0}^{u=1} dv$$

$$= \frac{1}{2}\int_0^{2\pi} \sin^2(v) dv$$

$$= \frac{\pi}{2}$$



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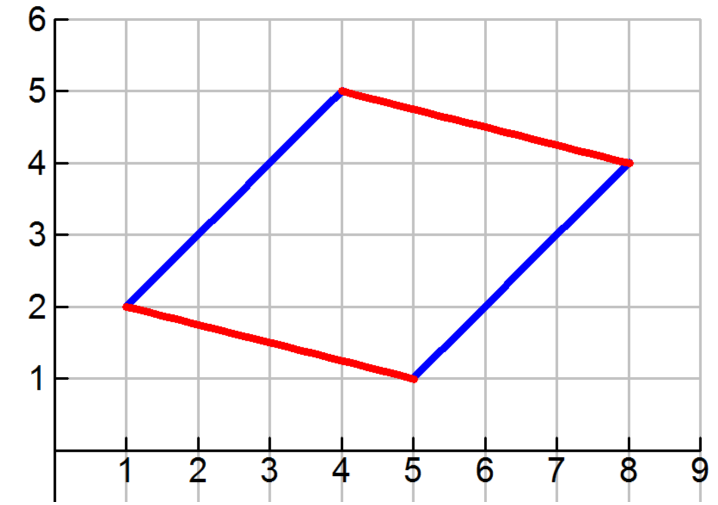
Example 7: Mathematica-Aided Change of Variables (Parallelogram Region)

Use Mathematica to compute $\iint_R e^y \, dA$ for R given by the parallelogram:



These lines are given by $y = x + 1, y = x - 4,$

$y = -\frac{1}{4}x + \frac{13}{4}$, and $y = -\frac{1}{4}x + 6$.

We can rewrite them as $1 = y - x, -4 = y - x,$

$\frac{1}{4}x + y = \frac{13}{4}$, and $\frac{1}{4}x + y = 6$.

Then we can let $u = y - x$ and $v = \frac{1}{4}x + y$ for $-4 \le u \le 1$ and $\frac{13}{4} \le v \le 6$!

But to compute our integral, we need the map from uv-space to xy-space...

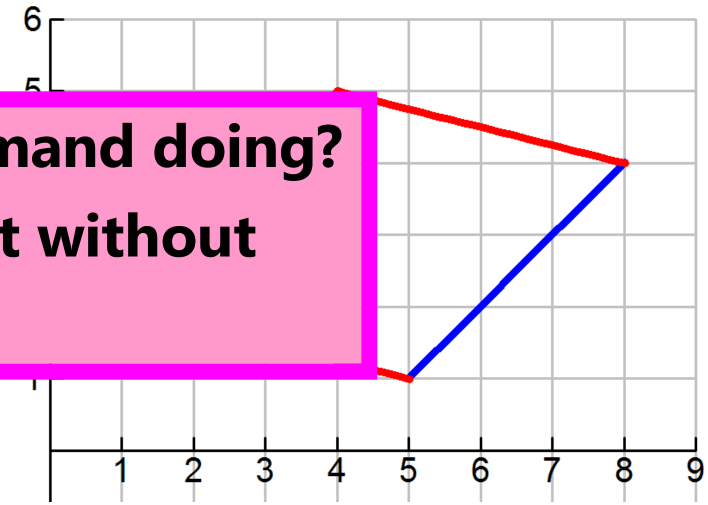
That is, we have $u(x, y)$ and $v(x, y)$, but we need $x(u, v)$ and $y(u, v)$.

Let Mathematica do the work: Solve[{$u = u[x, y], v = v[x, y]$},{x, y}]

$$x = -\frac{4}{5}(u - v) \text{ and } y = \frac{1}{5}(u + 4v)$$

Example 7: Mathematica-Aided Change of Variables (Parallelogram Region)

Use Mathematica to compute $\iint\limits_R e^y \, dA$ for R given by the parallelogram:



These lines are given by $y = x + 1$, $y = x - 4$,

$y = -\frac{1}{4}x + \frac{13}{4}$ …

We can rewr…

$\frac{1}{4}x + y = \frac{13}{4}$, and $\frac{1}{4}x + y = 6$.

What is this Mathematica command doing?
What will YOU do on a quiz/test without Mathematica?

Then we can let $u = y - x$ and $v = \frac{1}{4}x + y$ for $-4 \le u \le 1$ and $\frac{13}{4} \le v \le 6$!

But to compute our integral, we need the map from uv-space to xy-space…

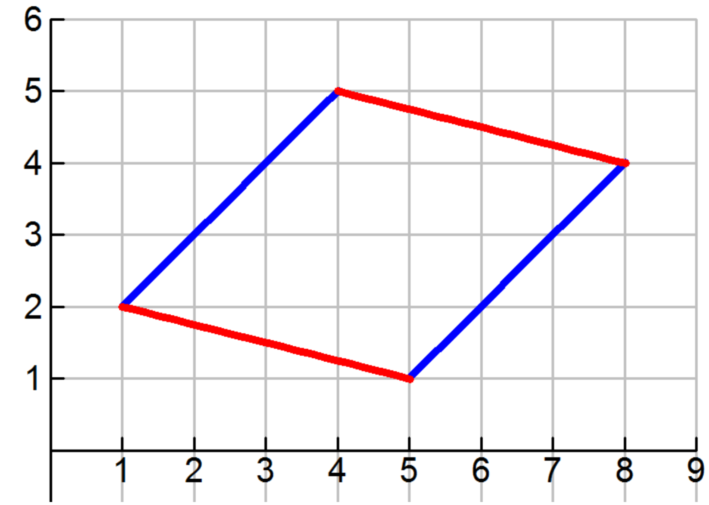
That is, we have $u(x,y)$ and $v(x,y)$, but we need $x(u,v)$ and $y(u,v)$.

Let Mathematica do the work: Solve[{$u = u[x,y]$, $v = v[x,y]$}, {x, y}]

$$x = -\frac{4}{5}(u - v) \text{ and } y = \frac{1}{5}(u + 4v)$$

Example 7: Mathematica-Aided Change of Variables (Parallelogram Region)

Use Mathematica to compute $\iint_R e^y \, dA$ for R given by the parallelogram:



$$x(u,v) = -\frac{4}{5}(u - v) \text{ and } y(u,v) = \frac{1}{5}(u + 4v)$$

$$A_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{4}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{vmatrix} = -\frac{4}{5}$$

So we use $\frac{4}{5}$

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Example 7: Mathema… of Variables (Parall…

Use Mathematica to compute $\iint_R e^y \, dA$ fo… :

$x(u,v) = -\frac{4}{5}(u-v)$ and $y(u,v) = \frac{1}{5}(u+4v)$

$$A_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{4}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{\ldots} \end{vmatrix} = -\frac{4}{5}$$

If $x(u,v) = -\frac{4}{5}(u-v)$, then

$$\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right) = \nabla x = \text{gradx}[u,v].$$

If $y(u,v) = \frac{1}{5}(u+4v)$, then

$$\left(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}\right) = \nabla y = \text{grady}[u,v].$$

So you can think of the area conversion factor as a determinant of a pair of gradient vectors from x(u,v) and y(u,v).

So we use $\frac{4}{5}$

Again, you can see that our change of variables is found by parameterizing our region (in this case, a parallelogram).

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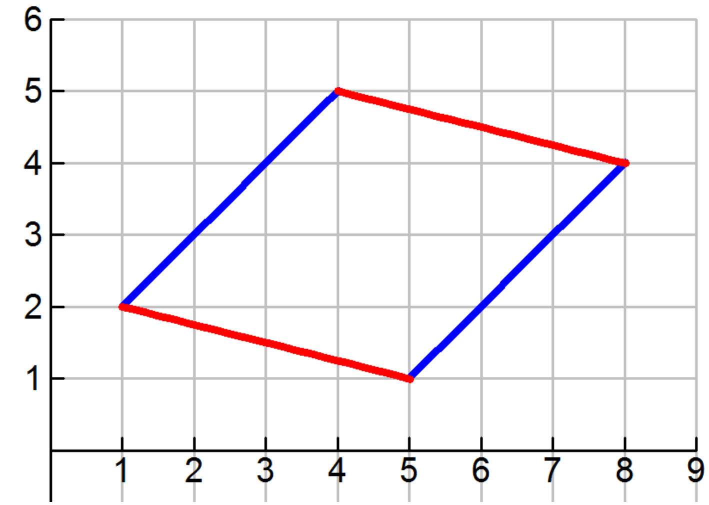
Example 7: Mathematica-Aided Change of Variables (Parallelogram Region)

Use Mathematica to compute $\iint_R e^y \, dA$ for R given by the parallelogram:

$$\left|A_{xy}(u,v)\right| = \frac{4}{5}, \; x = -\frac{4}{5}(u-v), \; y = \frac{1}{5}(u+4v),$$

for $-4 \le u \le 1$ and $\frac{13}{4} \le v \le 6$:



$$\iint_R e^y \, dA = \int_{13/4}^{6} \int_{-4}^{1} e^{(u+4v)/5} \left(\frac{4}{5}\right) du\, dv$$

$$= 417.1 \quad \text{(from Mathematica)}$$

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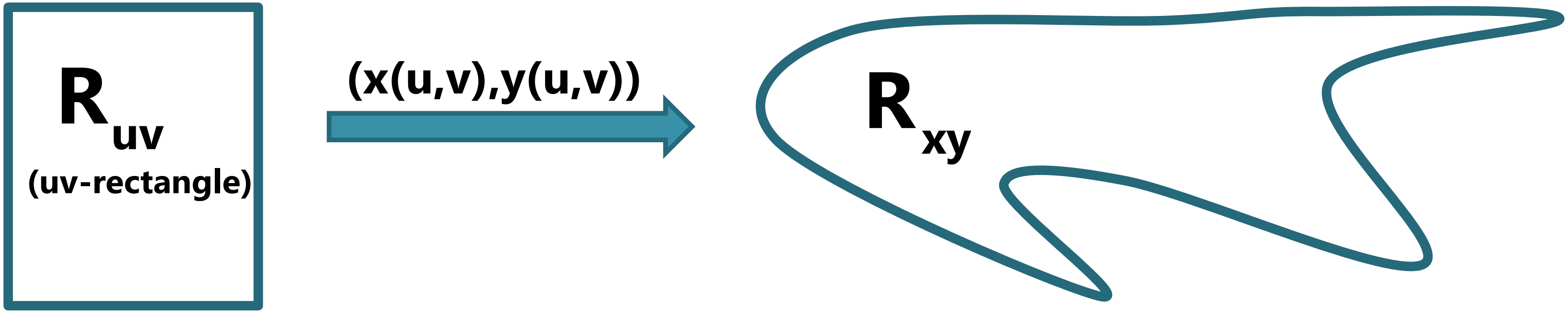
An Important Connection

A parameterization (x(u,v),y(u,v)) for a region R_{xy} with $a \le u \le b$ and $c \le v \le d$ is actually a map between a uv-rectangle, R_{uv}, and your messy xy-region, R_{xy}:



Long story short:

A parameterization IS a change of coordinates.

If you can parameterize a region, you can (probably) integrate over it.

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